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Einstein–Kähler metrics on a class of bundles involving integral weights

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Abstract

We prove that some compact complex bundles, defined over $\mathbb{P}_{d_1-1} \times \cdots \times \mathbb{P}_{d_n-1}$ (where $\mathbb{P}_d$ is the complex projective space of complex dimension d), and depending on n integral weights $a_1, \ldots, a_n$, have positive first Chern class if $1 \leqslant a_h \leqslant d_h - 1$ for all h, and carry Einstein–Kähler metrics when $a_1 = \cdots = a_n$ and $d_1 = \cdots = d_n$. © 2002 Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

Résumé

On montre que certains fibrés complexes compacts, définis au-dessus de $\mathbb{P}_{d_1-1} \times \cdots \times \mathbb{P}_{d_n-1}$ (où $\mathbb{P}_d$ désigne l'espace projectif complexe de dimension complexe d), et dépendant de n puissances entières $a_1, \ldots, a_n$, sont à première classe de Chern positive si $1 \leqslant a_h \leqslant d_h - 1$ pour tout h, et admettent une métrique d'Einstein–Kähler quand $a_1 = \cdots = a_n$ et $d_1 = \cdots = d_n$. © 2002 Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

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1. Introduction and main results

In this article, generalizing the class of bundles that we studied in [11,12], we give examples of compact Kähler manifolds with positive first Chern class which carry Einstein–Kähler metrics. We hope these examples will allow a better understanding of the case of more general algebraic manifolds. The problem of Einstein–Kähler metrics is described and studied [1,5,17,24]; for a general survey, see [15]. As regards original papers, concerning existence theorems, we refer to [2,3,21,23,26,28], and, for obstructions, to [16,19,20].

We investigate here projective bundles over the basis

$$B = B_{d_1,\ldots,d_n} = \mathbb{P}_{d_1-1} \times \cdots \times \mathbb{P}_{d_n-1} \quad (n \geqslant 2),$$

where $\mathbb{P}_k$ is the complex projective space of complex dimension k. These manifolds, which depend on integral weights $a_1, a_2, \ldots, a_n$, are labelled $X_{[d],[a]}$ and defined as follows:

Let $[d] = (d_1, \ldots, d_n)$ and $[a] = (a_1, \ldots, a_n)$ belong to $\mathbb{N}^n$, with $1 \leqslant a_h \leqslant d_h - 1$ for all $h = 1, \ldots, n$, and let $b_h = d_1 + d_2 + \cdots + d_h - 1$ and $m = b_n$. We also set

$$I_1 = \{0, \ldots, b_1\}, \qquad I_l = \{b_{l-1} + 1, \ldots, b_l\} \quad \text{for } l \geqslant 2,$$

and, if $Z_h = (z_k)_{k \in I_h} \in \mathbb{C}^{d_h}$,

$$Z_h^{a_h} = \left(z_k^{a_h}\right)_{k \in I_h}.$$

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Using these notations, $X_{[d],[a]}$ is the manifold

$$X_{[d],[a]} = \big\{p = \big([Z_1], \ldots, [Z_n], \big[\lambda_1 Z_1^{a_1}, \ldots, \lambda_n Z_n^{a_n}\big]\big) \in \mathbb{P}_{d_1-1} \times \cdots \times \mathbb{P}_{d_n-1} \times \mathbb{P}_m, \\ \text{where } \Lambda = [\lambda_1, \ldots, \lambda_n] \in \mathbb{P}_{n-1}\big\}. \tag{1}$$

$X = X_{[d],[a]}$ is a complex m-dimensional submanifold of $B \times \mathbb{P}_m$, and a complex projective bundle over B with fibers isomorphic to $\mathbb{P}_{n-1}$.

Let us now state the main results of this paper. The proof of Theorem 1 (respectively Theorem 2) is given in Section 2.3 (respectively Section 3).

Theorem 1. *The first Chern class $C_1(X)$ of X is positive if and only if $1 \leqslant a_h \leqslant d_h - 1$ for all $h = 1, \ldots, n$.*

Theorem 2. *Suppose that $d_1 = \cdots = d_n = d$, and $a_1 = \cdots = a_n = a$ with $1 \leqslant a \leqslant d - 1$. Then X admits an Einstein–Kähler metric.*

Under the conditions of Theorem 2, if G denotes the automorphisms group of X defined in 2.2, Tian's invariant $\alpha_G(X)$ is equal to one. When the dimensions d_h are distinct, as well as the weights a_h, one uses a different method to compute $\alpha_G(X)$. It involves Fano manifolds which generalize bundles introduced by Calabi, and it is studied in [13].

2. Geometry of the bundles $X_{[d],[a]}$

2.1. Description of charts and parametrizations of X

(**a**) The natural coordinates systems on the projective spaces $\mathbb{P}_{d_1-1}, \ldots, \mathbb{P}_{d_n-1}, \mathbb{P}_{n-1}$ generate an atlas of $X = X_{[d],[a]}$ with $nd_1 \ldots d_n$ charts whose domains are open dense subsets.

Let us describe one of them, labelled (U_0, φ_0). Its domain U_0 is the set of points

$$p = \big([z_0, \ldots, z_{b_1}], \ldots, [z_{b_{n-1}+1}, \ldots, z_m], \big[\lambda_1 z_0^{a_1}, \ldots, \lambda_1 z_{b_1}^{a_1}; \ldots; \lambda_n z_{b_{n-1}+1}^{a_n}, \ldots, \lambda_n z_m^{a_n}\big]\big) \in X$$

such that $z_0 \neq 0, z_{b_1+1} \neq 0, \ldots, z_{b_{n-1}+1} \neq 0, \lambda_1 \neq 0$; hence, the first components of the vectors $Z_1, \ldots, Z_n$, Λ occurring in the description of p given in Definition (1) of Section 1 are different from zero. Now, let us set

$$\zeta_1 = \frac{z_1}{z_0}, \quad \ldots, \quad \zeta_{b_1} = \frac{z_{b_1}}{z_0}; \qquad \zeta_{b_1+1} = \frac{\lambda_2 z_{b_1+1}^{a_2}}{\lambda_1 z_0^{a_1}}, \qquad \zeta_{b_1+2} = \frac{z_{b_1+2}}{z_{b_1+1}}, \quad \ldots, \quad \zeta_{b_2} = \frac{z_{b_2}}{z_{b_1+1}}; \quad \ldots;$$
$$\zeta_{b_{n-1}+1} = \frac{\lambda_n z_{b_{n-1}+1}^{a_n}}{\lambda_1 z_0^{a_1}}, \qquad \zeta_{b_{n-1}+2} = \frac{z_{b_{n-1}+2}}{z_{b_{n-1}+1}}, \quad \ldots, \quad \zeta_m = \frac{z_m}{z_{b_{n-1}+1}}.$$

Then,

$$p = \big([1, \zeta_1, \ldots, \zeta_{b_1}], \ldots, [1, \zeta_{b_{n-1}+2}, \ldots, \zeta_m], \big[1, \zeta_1^{a_1}, \ldots, \zeta_{b_1}^{a_1}; \zeta_{b_1+1}\big(1, \zeta_{b_1+2}^{a_2}, \ldots, \zeta_{b_2}^{a_2}\big); \ldots; \\ \zeta_{b_{n-1}+1}\big(1, \zeta_{b_{n-1}+2}^{a_n}, \ldots, \zeta_m^{a_n}\big)\big]\big)$$

and φ_0 is the one-to-one mapping from U_0 into $\mathbb{C}^m$ given by

$$\varphi_0(p) = (\zeta_1, \ldots, \zeta_m).$$

The other charts of the atlas are obtained in an analogous way by assigning to each vector $Z_1, \ldots, Z_n, \Lambda$ a component different from 0 which can thus be taken equal to 1.

(**b**) Let V be the subset of X such that all the components of the vectors $Z_1, \ldots, Z_n$ and Λ occurring in Definition (1) of Section 1 are different from zero. We now describe several parametrizations ψ_h $(h = 1, \ldots, n)$ of V by the open set of $\mathbb{C}^m$:

$$\mathbb{C}_*^m = \big\{(z_k)_{1 \leqslant k \leqslant m} \in \mathbb{C}^m;\ z_k \neq 0 \text{ for all } k\big\}.$$

Let us choose $h \in \{1, \ldots, n\}$. For any $j = 1, \ldots, n$, we pick $Z_j = (z_k)_{k \in I_j} \in \mathbb{C}^{d_j}$ with $z_k \neq 0$ for all $k \in I_j$, and we suppose that the first component $z_{b_{h-1}+1}$ of Z_h is equal to one (we could impose this condition to any other component of Z_h). We identify $Z = (Z_1, \ldots, Z_n)$ with the point $(z_k)_{0 \leqslant k \leqslant m,\, k \neq b_{h-1}+1}$ of $\mathbb{C}_*^m$ and we set

$$\psi_h(Z) = \big([Z_1], \ldots, [Z_n], \big[Z_1^{a_1}, \ldots, Z_n^{a_n}\big]\big) \in V,$$

ψ_h is a surjective mapping from $\mathbb{C}_*^m$ onto V. Suppose that $p \in V$ is such that $p = \psi_h(Z) = \psi_h(Z')$, with Z and $Z' \in \mathbb{C}_*^m$. Then, $z_k = z'_k$ when $k \in I_h$, and for any $j \neq h$, there exists ν_j such that

$$z'_k = \nu_j z_k \quad \text{and} \quad {z'_k}^{a_j} = (\nu_j z_k)^{a_j} \quad \text{for all } k \in I_j.$$

Consequently, $\nu_j^{a_j} = 1$ and ν_j is an a_j-th root of the unity in $\mathbb{C}$. Hence, any $p \in V$ is the image by ψ_h of

$$a'_h = \prod_{j=1,\ j\neq h}^{n} a_j$$

elements of $\mathbb{C}_*^m$. ψ_h is a parametrization of V by $\mathbb{C}_*^m$ which covers a'_h times V. To obtain a chart of V, we pick, for any $j \in \{1, \dots, n\}$ different from h, an index $k_j \in I_j$, and we impose that the argument of the component z_{k_j} of Z_j belongs to some fixed interval of length $(2\pi/a_j)$.

2.2. *Automorphisms group of X*

(**a**) To define a group G of automorphisms of $X_{[d],[a]}$, we use the automorphisms groups of $\mathbb{P}_{d_1-1}, \dots, \mathbb{P}_{d_n-1}$ obtained by multiplication by $e^{i\theta}$ ($\theta \in \mathbb{R}$) and permutation of the homogeneous coordinates. Indeed, if σ is such an automorphism of $\mathbb{P}_{d_h-1}$, it induces a transformation of X which maps

$$\bigl([Z_1], \dots, [Z_h], \dots, [Z_n], \bigl[\lambda_1 Z_1^{a_1}, \dots, \lambda_n Z_n^{a_n}\bigr]\bigr) \in \mathbb{P}_{d_1-1} \times \cdots \times \mathbb{P}_{d_n-1} \times \mathbb{P}_m,$$

into

$$\bigl([Z_1], \dots, \bigl[\sigma(Z_h)\bigr], \dots, [Z_n], \bigl[\lambda_1 Z_1^{a_1}, \dots, \lambda_h \bigl(\sigma(Z_h)\bigr)^{a_h}, \dots, \lambda_n Z_n^{a_n}\bigr]\bigr).$$

If $d_h = d_k$ and $a_h = a_k = a$ (with $1 \leqslant h < k \leqslant n$), we also consider the automorphism of X induced by permutation of Z_h and Z_k; it is defined as follows:

$$\begin{aligned}&\bigl([Z_1], \dots, [Z_h], \dots, [Z_k], \dots, [Z_n], \bigl[\lambda_1 Z_1^{a_1}, \dots, \lambda_h Z_h^{a_h}, \dots, \lambda_k Z_k^{a_k}, \dots, \lambda_n Z_n^{a_n}\bigr]\bigr)\\ &\quad\to \bigl([Z_1], \dots, [Z_k], \dots, [Z_h], \dots, [Z_n], \bigl[\lambda_1 Z_1^{a_1}, \dots, \lambda_k Z_k^{a_k}, \dots, \lambda_h Z_h^{a_h}, \dots, \lambda_n Z_n^{a_n}\bigr]\bigr).\end{aligned}$$

The group of automorphisms of X generated by the previous ones will be denoted by G. Notice that the the dense open subset V of X defined in 2.1(b) is G-invariant.

(**b**) Now, let $\varphi \in C^\infty(X)$ be a G-invariant function. We want to examine the effect of this invariance on the expressions $\varphi \circ \psi_h = \varphi_h$ (defined on $\mathbb{C}_*^m$) of φ in the parametrizations $(\psi_h)_{1\leqslant h\leqslant n}$ of V. Suppose, to simplify the notations, that $h = 1$ and write

$$\varphi_1 = \varphi_1(1, z_1, \dots, z_m) = \varphi\bigl([Z_1], \dots, [Z_n], \bigl[Z_1^{a_1}, \dots, Z_n^{a_n}\bigr]\bigr) = \varphi(p),$$

where $Z_1 = (1, z_1, \dots, z_{b_1})$ considered as belonging to $\mathbb{C}_*^{d_1-1}$ and $Z_j = (z_k)_{k\in I_j} \in \mathbb{C}_*^{d_j}$ if $j \geqslant 2$.

(**b.1**) First, φ_1 is $\tau_{k,\theta}$-invariant, i.e. invariant by multiplication of the z_k by any $e^{i\theta}$ ($\theta \in \mathbb{R}$). Hence, it depends only on the $x_k = |z_k|^2$, and we consider φ_1 as a function of $(x_1, \dots, x_m) \in \mathbb{R}_*^m$. Then φ_1 is invariant by permutation of any x_k, x_l (if $1 \leqslant k < l \leqslant n$ and (k, l) belong to the same subset I_j).

(**b.2**) Now, we establish the link between φ_1 and φ_h (when $h = 2$ for instance), using only the invariance by the automorphisms $\tau_{j,\theta}$. If $p \in V$, we consider the following two manners of describing p:

$$p = \bigl([Z_1] = [1, z_1, \dots, z_{b_1}], [Z_2] = [z_{b_1+1}, \dots, z_{b_2}], [Z_3], \dots, [Z_n], \bigl[Z_1^{a_1}, Z_2^{a_2}, \dots, Z_n^{a_n}\bigr]\bigr),$$

and

$$p = \bigl([Z'_1] = \bigl[z'_0, z'_1, \dots, z'_{b_1}\bigr], \bigl[Z'_2\bigr] = \bigl[1, z'_{b_1+2}, \dots, z'_{b_2}\bigr], \bigl[Z'_3\bigr], \dots, \bigl[Z'_n\bigr], \bigl[{Z'_1}^{a_1}, {Z'_2}^{a_2}, \dots, {Z'_n}^{a_n}\bigr]\bigr).$$

From these representations, we deduce that, from some complex numbers $\nu_1, \dots, \nu_n, \nu \neq 0$, we have

$$Z'_h = \nu_h Z_h \quad \text{and} \quad \bigl({Z'_1}^{a_1}, \dots, {Z'_n}^{a_n}\bigr) = \nu\bigl(Z_1^{a_1}, \dots, Z_n^{a_n}\bigr).$$

Thus,

$$z'_0 = \nu_1, \qquad 1 = \nu_2 z_{b_1+1}, \qquad {z'_0}^{a_1} = \nu, \qquad 1 = \nu z_{b_1+1}^{a_2},$$

which implies

$$\nu = z_{b_1+1}^{-a_2}, \qquad \nu_1 = z'_0 = \nu^{1/a_1} = z_{b_1+1}^{-a_2/a_1}, \qquad \nu_2 = z_{b_1+1}^{-1}$$

and, if $h \geqslant 3$, $\nu_h = z_{b_1+1}^{-a_2/a_h}$ since, for $k \in I_h$,

$$z'_k = \nu_h z_k, \qquad {z'_k}^{a_h} = \nu z_k^{a_h} = \nu_h^{a_h} z_k^{a_h}, \quad \text{i.e.} \quad \nu_h = \nu^{1/a_h}.$$

The relation between φ_1 and φ_2 follows:

$$\begin{aligned}\varphi(p) &= \varphi_1(1, x_1, \dots, x_m)\\ &= \varphi_2\left(\frac{1}{x_{b_1+1}^{a_2/a_1}}, \frac{x_1}{x_{b_1+1}^{a_2/a_1}}, \dots, \frac{x_{b_1}}{x_{b_1+1}^{a_2/a_1}}; 1, \frac{x_{b_1+2}}{x_{b_1+1}}, \dots, \frac{x_{b_2}}{x_{b_1+1}}; \frac{x_{b_2+1}}{x_{b_1+1}^{a_2/a_3}}, \dots, \frac{x_{b_3}}{x_{b_1+1}^{a_2/a_3}}; \dots; \frac{x_{b_{n-1}+1}}{x_{b_1+1}^{a_2/a_n}}, \dots, \frac{x_m}{x_{b_1+1}^{a_2/a_n}}\right).\end{aligned}$$

(**b.3**) In an analogous way, let us express that φ_1 is invariant by permutation $\sigma_{0,1}$of the homogeneous coordinates z_0 and z_1 of Z_1. Notice that

$$p' = \sigma_{0,1}(p) = \big([z_1, 1, z_2, \dots, z_n], [Z_2], \dots, [Z_n], \big[z_1^{a_1}, 1, z_2^{a_1}, \dots, z_{b_1}^{a_1}, Z_2^{a_2}, \dots, Z_n^{a_n}\big]\big).$$

To compute $\varphi_1(p')$, we must represent p' under the following form:

$$p' = \big(\big[1, z'_1, \dots, z'_{b_1}\big], \big[Z'_2\big], \dots, \big[Z'_n\big], \big[1, {z'_1}^{a_1}, {z'_2}^{a_1}, \dots, {z'_{b_1}}^{a_1}, {Z'_2}^{a_2}, \dots, {Z'_n}^{a_n}\big]\big),$$

with $Z'_h \in \mathbb{C}_*^{d_h}$. So we get

$$z'_1 = \frac{1}{z_1}, \qquad z'_2 = \frac{z_2}{z_1}, \quad \dots, \quad z'_{b_1} = \frac{z_{b_1}}{z_1}$$

and, if $h \geqslant 2$, $k \in I_h$, for some $\nu \neq 0$,

$$z'_k = \nu_h z_k, \qquad {z'_k}^{a_h} = \frac{z_k^{a_h}}{z_1^{a_1}}.$$

Hence, $\nu_h^{a_h} = 1/z_1^{a_1}$, i.e. $\nu_h = z_1^{-a_1/a_h}$ for some determination of the power $z_1^{-a_1/a_h}$ which has not to be precised since we only consider $x_k = |z_k|^2$, $x'_k = |z'_k|^2$, and $x'_k = x_k/x_1^{a_1/a_h}$.

Consequently, we obtain

$$\varphi_1(1, x_1, \dots, x_m) = \varphi_1\left(1, \frac{1}{x_1}, \frac{x_2}{x_1}, \dots, \frac{x_{b_1}}{x_1}; \frac{x_{b_1+1}}{x_1^{a_1/a_2}}, \dots, \frac{x_{b_2}}{x_1^{a_1/a_2}}; \dots; \frac{x_{b_{n-1}+1}}{x_1^{a_1/a_n}}, \dots, \frac{x_m}{x_1^{a_1/a_n}}\right).$$

(**b.4**) Finally, if $d_1 = d_2 = d$ and $a_1 = a_2 = a$, and if σ is the automorphism which permutes $[Z_1]$ and $[Z_2]$, taking into account the invariance of φ by σ and the $\tau_{j,\theta}$ yields:

$$\begin{aligned}\varphi_1(1, x_1, \dots, x_m) &= \varphi(p) = \varphi\big(\sigma(p)\big) = \varphi\big([Z_2], [Z_1], [Z_3], \dots, [Z_n], \big[Z_2^a, Z_1^a, Z_3^{a_3}, \dots, Z_n^{a_n}\big]\big)\\ &= \varphi_2(x_d, \dots, x_{2d-1}; 1, x_1, \dots, x_{d-1}; x_{2d}, \dots, x_m)\\ &= \varphi_1\left(1, \frac{x_{d+1}}{x_d}, \dots, \frac{x_{2d-1}}{x_d}; \frac{1}{x_d}, \frac{x_1}{x_d}, \dots, \frac{x_{d-1}}{x_d}; \frac{x_{2d}}{x_d^{a/a_3}}, \dots, \frac{x_{2d+d_3-1}}{x_d^{a/a_3}}; \dots; \frac{x_{b_{n-1}+1}}{x_d^{a/a_n}}, \dots, \frac{x_m}{x_d^{a/a_n}}\right).\end{aligned}$$

2.3. *First Chern class and metric of X*

(**a**) If one uses the atlas defined in 2.1(a), there are two generic types of changes of coordinates.

Let us start with charts (α) parametrizing points

$$p = \big([Z_1], \dots, [Z_n], \big[\lambda_1 Z_1^{a_1}, \dots, \lambda_n Z_n^{a_n}\big]\big) \in X$$

such that the first components of $Z_1, \dots, Z_n$ and Λ are equal to one; the (α) coordinates of p are

$$z_1, \dots, z_{b_1}; \lambda_2, z_{b_1+2}, \dots, z_{b_2}; \dots; \lambda_n, z_{b_{n-1}+1}, \dots, z_m.$$

To change the position of the components equal to one in $Z_1,\dots,Z_n$ and Λ, we proceed step by step, changing successively in Z_1, then in $Z_2,\dots,Z_n$ and finally in Λ (the roles of Z_h being symmetric).

On the first hand, we consider chart (β) which differs from (α) by the fact that the second component of Z_1 is now equal to one. The corresponding change of coordinates $\Gamma_{\beta,\alpha}$ from chart (α) to chart (β) maps each point $(z_1,\dots,z_m)$, with $z_1\neq 0$ to point

$$\left(\frac{1}{z_1},\frac{z_2}{z_1},\dots,\frac{z_{b_1}}{z_1};\frac{z_{b_1+1}}{z_1^{a_1}},z_{b_1+2},\dots,z_{b_2};\dots;\frac{z_{b_{n-1}+1}}{z_1^{a_1}},z_{b_{n-1}+2},\dots,z_m\right),$$

as is seen if we express, for instance when $n=2$, the following equality:

$$\begin{aligned}&\big([1,z_1,\dots,z_{b_1}],[1,z_{b_1+2},\dots,z_m],\big[1,z_1^{a_1},\dots,z_{b_1}^{a_1},z_{b_1+1}\big(1,z_{b_1+2}^{a_2},\dots,z_m^{a_2}\big)\big]\big)\\&\quad=\big([u_0,1,u_2,\dots,u_{b_1}],[1,u_{b_1+2},\dots,u_m],\big[u_0^{a_1},1,u_2^{a_1},\dots,u_{b_1}^{a_1},u_{b_1+1}\big(1,u_{b_1+2}^{a_2},\dots,u_m^{a_2}\big)\big]\big).\end{aligned}$$

Let us compute the Jacobian $J_{\beta,\alpha}$ of $\Gamma_{\beta,\alpha}$. It is given by

$$J_{\beta,\alpha}=\frac{1}{z_1^{d_1+(n-1)a_1}}.$$

On the other hand, let us denote by (γ) the chart for which condition $\lambda_1=1$ of chart (α) becomes $\lambda_2=1$. The change of coordinates from chart (α) to chart (γ) is given by

$$\begin{aligned}&\Gamma_{\gamma,\alpha}:(z_1,\dots,z_{b_1+1}\neq 0,\dots,z_m)\\&\quad\to\left(\frac{1}{z_{b_1+1}},z_1,\dots,z_{b_1};z_{b_1+2},\dots,z_{b_2};\frac{z_{b_2+1}}{z_{b_1+1}},z_{b_2+2},\dots,z_{b_3};\dots;\frac{z_{b_{n-1}+1}}{z_{b_1+1}},z_{b_{n-1}+2},\dots,z_m\right),\end{aligned}$$

as we obtain if we consider (for $n=2$) the following two representations of $p\in X$, which yield the coordinates of p in charts (α) and (γ):

$$\begin{aligned}p&=\big([1,z_1,\dots,z_{b_1}],[1,z_{b_1+2},\dots,z_m],\big[1,z_1^{a_1},\dots,z_{b_1}^{a_1},z_{b_1+1}\big(1,z_{b_1+2}^{a_2},\dots,z_m^{a_2}\big)\big]\big)\\&=\big([1,z_1,\dots,z_{b_1}],[1,z_{b_1+2},\dots,z_m],\big[z_0\big(1,z_1^{a_1},\dots,z_{b_1}^{a_1}\big);1,z_{b_1+2}^{a_2},\dots,z_m^{a_2}\big]\big).\end{aligned}$$

The Jacobian $J_{\gamma,\alpha}$ of $\Gamma_{\gamma,\alpha}$ is equal to

$$J_{\gamma,\alpha}=\frac{1}{z_{b_1+1}^n}.$$

(**b**) We now look for an element $\omega\in C_1(X)$.

Let us consider a chart $(U_\delta,\varphi_\delta)$, labelled (δ) (its domain U_δ is an open subset of X containing V and φ_δ is an isomorphism between U_δ and $\mathbb{C}^m$), of the atlas defined in 2.1(a) which corresponds to some choice of a component equal to one for each vector $Z_1,\dots,Z_n$ and Λ occurring in the description of $p=([Z_1],\dots,[Z_n],[\lambda_1 Z_1^{a_1},\dots,\lambda_n Z_n^{a_n}])\in X$ (see Definition (1) in Section 1).

In chart (δ), we seek ω as $\mathrm{i}\partial\bar\partial K_\delta$ where the potential K_δ is defined by $K_\delta=\log E_\delta$, with

$$E_\delta(p)=|Z_1|^{2r_1}\cdots|Z_n|^{2r_n}\big(l_1\big|Z_1^{a_1}\big|^2+\cdots+l_n\big|Z_n^{a_n}\big|^2\big)^q.$$

Here for any $Z=(z_1,\dots,z_s)\in\mathbb{C}^s$, $\Lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n$ and $a\in\mathbb{N}$, we set

$$|Z|^2=\sum_{j=1}^s x_j,\quad Z^a=\big(z_1^a,\dots,z_s^a\big),\ x_j=|z_j|^2\ \text{and}\ l_j=|\lambda_j|^2.$$

This choice is natural if we consider the pull-back on X of convenient multiples of the Fubini–Study metric by the natural projections of $X\subset\mathbb{P}_{d_1-1}\times\cdots\times\mathbb{P}_{d_n-1}\times\mathbb{P}_m$ on the factors $\mathbb{P}_{d_1-1},\dots,\mathbb{P}_{d_n-1}$ and $\mathbb{P}_m$.

We try to find the exponents $r_1,\dots,r_n,q$ such that the functions E_δ can be viewed as the local expressions of an Hermitian metric on the determinant line bundle of X. Consequently the local functions E_δ must satisfy the following compatibility relations: $E_\beta=|J_{\beta,\alpha}|^2E_\alpha$, if we consider for instance charts (α) and (β). Under these conditions, the curvature form $\omega=\mathrm{i}\partial\bar\partial\log E_\delta$ is independant of chart (δ) and represents a well defined 1-1 form on X which belongs to $C_1(X)$.

Let $z_1,\dots,z_m$ be the (α)-coordinates of $p\in U_\alpha$. If $p\in U_\alpha\cap U_\gamma$, computing the value of $E_\gamma(p)$ in terms of the coordinates of p in chart (α), we get:

$$
\begin{aligned}
E_\gamma(p) \;=\;& (1+x_1+\cdots+x_{b_1})^{r_1}(1+x_{b_1+2}+\cdots+x_{b_2})^{r_2}\cdots(1+x_{b_{n-1}+2}+\cdots+x_m)^{r_n}\\
&\times\biggl[\frac{1}{x_{b_1+1}}\bigl(1+x_1^{a_1}+\cdots+x_{b_1}^{a_1}\bigr)+\bigl(1+x_{b_1+2}^{a_2}+\cdots+x_{b_2}^{a_2}\bigr)+\frac{x_{b_2+1}}{x_{b_1+1}}\bigl(1+x_{b_2+2}^{a_3}+\cdots+x_{b_3}^{a_3}\bigr)\\
&\quad+\cdots+\frac{x_{b_{n-1}+1}}{x_{b_1+1}}\bigl(1+x_{b_{n-1}+2}^{a_n}+\cdots+x_m^{a_n}\bigr)\biggr]^q\\
=\;& \frac{1}{x_{b_1+1}^q}E_\alpha(p).
\end{aligned}
$$

Thus, since $E_\gamma = |J_{\gamma,\alpha}|^2 E_\alpha$, taking into account the value of $J_{\gamma,\alpha}$ that we obtained in paragraph (a), we must have

$$\frac{1}{x_{b_1+1}^q} = |J_{\gamma,\alpha}|^2 = \frac{1}{x_{b_1+1}^n},$$

that is $q=n$.

On the other hand, if $p\in U_\alpha\cap U_\beta$, we see that

$$
\begin{aligned}
E_\beta(p) \;=\;& \left(\frac{1+x_1+\cdots+x_{b_1}}{x_1}\right)^{r_1}(1+x_{b_1+2}+\cdots+x_{b_2})^{r_2}\cdots(1+x_{b_{n-1}+2}+\cdots+x_m)^{r_n}\\
&\times\biggl[\frac{1+x_1^{a_1}+\cdots+x_{b_1}^{a_1}}{x_1^{a_1}}+\frac{x_{b_1+1}}{x_1^{a_1}}\bigl(1+x_{b_1+2}^{a_2}+\cdots+x_{b_2}^{a_2}\bigr)+\cdots+\frac{x_{b_{n-1}+1}}{x_1^{a_1}}\bigl(1+x_{b_{n-1}+2}^{a_n}+\cdots+x_m^{a_n}\bigr)\biggr]^q\\
=\;& \frac{1}{x_1^{r_1+qa_1}}E_\alpha(p).
\end{aligned}
$$

Hence, we obtain

$$\frac{1}{x_1^{r_1+qa_1}} = |J_{\beta,\alpha}|^2 = \frac{1}{x_1^{d_1+(n-1)a_1}}$$

and, since $q=n$, $r_1+na_1=d_1+(n-1)a_1$, i.e. $r_1=d_1-a_1$. In fact, $r_h=d_h-a_h$ for $h=1,\dots,n$.

(**c**) And now, let us collect some properties of ω. First, thanks to the form of the potential $K=\log E$ (where we omit any reference to some chart), $\omega=\mathrm{i}\partial\overline{\partial}K$ is everywhere positive definite as we shall see later, so the first Chern class of X is positive. We shall use the corresponding Kähler metric g, written locally $g_{\lambda\overline{\mu}}=\partial_{\lambda\overline{\mu}}K$.

Let us check that ω is positive definite, in chart (α) for instance. In this chart, we write $\omega=\sum_{l=1}^{n+1}\omega_l$, with

$$
\begin{aligned}
\omega_1 \;&=\; \mathrm{i}(d_1-a_1)\partial\overline{\partial}\log(1+x_1+\cdots+x_{b_1}),\\
\omega_h \;&=\; \mathrm{i}(d_h-a_h)\partial\overline{\partial}\log(1+x_{b_{h-1}+2}+\cdots+x_{b_h})\quad \text{for } h=2,\dots,n,
\end{aligned}
$$

and

$$\omega_{n+1}=\mathrm{i}n\partial\overline{\partial}\log\Biggl[1+x_1^{a_1}+\cdots+x_{b_1}^{a_1}+\sum_{h=2}^{n}x_{b_{h-1}+1}\bigl(1+x_{b_{h-1}+2}^{a_h}+\cdots+x_{b_h}^{a_h}\bigr)\Biggr].$$

All these one-to-one forms are considered, at every point, as Hermitian forms on $\mathbb{C}^m$. Clearly, $\omega_1,\dots,\omega_n$ are non-negative since they correspond to multiples of the Fubini–Study metric on $\mathbb{C}^{d_1-1},\dots,\mathbb{C}^{d_n-1}$, respectively. ω_{n+1} is also non-negative since it is the pull-back of the Fubini–Study metric $\mathrm{i}n\partial\overline{\partial}\log(1+v_1^2+\cdots+v_m^2)$ on $\mathbb{C}^m$ by the following holomorphic map:

$$
\begin{aligned}
(z_1,\dots,z_m) \;&\to\; (v_1,\dots,v_m)\\
&=\; \bigl(z_1^{a_1},\dots,z_{b_1}^{a_1};z_{b_1+1},z_{b_1+1}z_{b_1+2}^{a_2},\dots,z_{b_1+1}z_{b_2}^{a_2};\dots;z_{b_{n-1}+1},z_{b_{n-1}+1}z_{b_{n-1}+2}^{a_n},\dots,z_{b_{n-1}+1}z_m^{a_n}\bigr).
\end{aligned}
$$

At any point $p=(z_1,\dots,z_m)$, we have

$$V(p) \;=\; \bigcap_{h=1}^{n}\operatorname{Ker}\omega_h(p)=\bigl\{\zeta=(\zeta_1,\dots,\zeta_m)\in\mathbb{C}^m;\ \zeta_k=0 \text{ for } k\neq b_1+1,\dots,b_{n-1}+1\bigr\}.$$

To prove that $\omega(p)$ is positive definite, one has to show that the restriction of $\omega_{n+1}(p)$ to $V(p)$ is itself positive definite. Suppressing the dependance in p, we set:

$$\begin{aligned} u_1 &= z_{b_1+1}, \dots, u_{n-1} = z_{b_{n-1}+1}, \\ S_1 &= 1 + x_1^{a_1} + \dots + x_{b_1}^{a_1}, \dots, S_h = 1 + x_{b_{h-1}+2}^{a_h} + \dots + x_{b_h}^{a_h} \quad \text{for } 2 \leqslant h \leqslant n-1, \end{aligned}$$

and

$$S = S_1\left[1 + \sum_{h=1}^{n-1}\left(u_h\sqrt{\frac{S_h}{S_1}}\right)\left(\bar{u}_h\sqrt{\frac{S_h}{S_1}}\right)\right] = S_1 T.$$

Notice that $S_1, \dots, S_{n-1}$ do not depend on $u = (u_1, \dots, u_{n-1})$.

Now, let $\zeta = (\zeta_1, \dots, \zeta_m) \in V$ and $\zeta_1^* = \zeta_{b_1+1}, \dots, \zeta_{n-1}^* = \zeta_{b_{n-1}+1}$. We have

$$\omega_{n+1}(\zeta, \bar{\zeta}) = n \sum_{\alpha,\beta=1}^{n-1} \frac{\partial^2 \log S}{\partial u_\alpha \partial \bar{u}_\beta} \zeta_\alpha^* \overline{\zeta_\beta^*} = n \sum_{\alpha,\beta=1}^{n-1} \frac{\partial^2 \log T}{\partial u_\alpha \partial \bar{u}_\beta} \zeta_\alpha^* \overline{\zeta_\beta^*}.$$

Hence, by virtue of the positive definiteness of the Fubini–Study metric on $\mathbb{C}^{n-1}$, $\omega_{n+1}(\zeta, \bar{\zeta}) = 0$ if and only if $\zeta_1^* = \dots = \zeta_{n-1}^* = 0$, i.e. $\zeta = 0$. From this, we deduce that the restriction of ω_{n+1} to V is positive definite, as requested.

(**d**) ω is G-invariant. In fact, for any generator τ of G defined in 2.2, if $p \in X$ and $p' = \tau(p)$, we can choose charts (δ) and (δ') whose domains also contain p and p' and such that $K_{\delta'} \circ \tau = K_\delta$ in the neighborhood of p. Thus,

$$\tau^*\omega = \tau^*\big(\mathrm{i}\partial\bar{\partial}K_{\delta'}\big) = \mathrm{i}\partial\bar{\partial}(K_{\delta'} \circ \tau) = \mathrm{i}\partial\bar{\partial}K_\delta = \omega.$$

Finally, we shall need the local expressions $\omega_h = \psi_h^*(\omega)$ of ω (or the corresponding metric g) in the parametrizations ψ_h of the open subset $V \subset X$ defined in 2.1(b). To simplify the notations, we take $h = 1$. Since $V \subset U_\alpha$, the mapping

$$\Gamma_{\alpha,1} = \varphi_\alpha \circ \psi_1 : \mathbb{C}_*^m \to \mathbb{C}^m,$$

which is an $a_1' = a_2 \cdots a_n$-fold covering, is given by

$$\Gamma_{\alpha,1} : Z = (z_1, \dots, z_m) \ \to \ \left(z_1, \dots, z_{b_1}, z_{b_1+1}^{a_2}, \frac{z_{b_1+2}}{z_{b_1+1}}, \dots, \frac{z_{b_2}}{z_{b_1+1}}; \dots; z_{b_{n-1}+1}^{a_n}, \frac{z_{b_{n-1}+2}}{z_{b_{n-1}+1}}, \dots, \frac{z_m}{z_{b_{n-1}+1}}\right),$$

as we see by writing

$$\begin{aligned} p &= \big([1, z_1, \dots, z_{b_1}], [z_{b_1+1}, \dots, z_{b_2}], \dots, [z_{b_{n-1}+1}, \dots, z_m], \\ &\qquad \big[1, z_1^{a_1}, \dots, z_{b_1}^{a_1}, z_{b_1+1}^{a_2}, \dots, z_{b_2}^{a_2}; \dots; z_{b_{n-1}+1}^{a_n}, \dots, z_m^{a_n}\big]\big) \\ &= \big([1, u_1, \dots, u_{b_1}], [1, u_{b_1+2}, \dots, u_{b_2}], \dots, [1, u_{b_{n-1}+2}, \dots, u_m], \\ &\qquad \big[1, u_1^{a_1}, \dots, u_{b_1}^{a_1}; u_{b_1+1}\big(1, u_{b_1+2}^{a_2}, \dots, u_{b_2}^{a_2}\big); \dots; u_{b_{n-1}+1}\big(1, u_{b_{n-1}+2}^{a_n}, \dots, u_m^{a_n}\big)\big]\big). \end{aligned}$$

Hence, if

$$t_1 = 1 + \sum_{k=1}^{b_1} x_k, \qquad t_h = \sum_{k=b_{h-1}+1}^{b_h} x_k \quad \text{for } h = 2, \dots, n \quad \text{and} \quad T = 1 + \sum_{k=1}^{b_1} x_k^{a_1} + \sum_{h=2}^{n} \sum_{k=b_{h-1}+1}^{b_h} x_k^{a_h},$$

we get

$$E_\alpha\big(\Gamma_{\alpha,1}(Z)\big) = t_1^{d_1-a_1}\left(\frac{t_2}{x_{b_1+1}}\right)^{d_2-a_2} \cdots \left(\frac{t_n}{x_{b_{n-1}+1}}\right)^{d_n-a_n} T^n.$$

Consequently, since $\partial\bar{\partial}\log(x_{b_1+1}^{d_2-a_2} \cdots x_{b_{n-1}+1}^{d_n-a_n}) = 0$, we obtain

$$\omega_1 = \Gamma_{\alpha,1}^*\big(\mathrm{i}\partial\bar{\partial}K_\alpha\big) = \mathrm{i}\partial\bar{\partial}(K_\alpha \circ \Gamma_{\alpha,1}) = \mathrm{i}\partial\bar{\partial}K_1,$$

where the local potential K_1 is defined by $K_1 = \log(T^n \prod_{h=1}^n t_h^{d_h-a_h})$.

2.4. Volume element of the metric g

We want to compute the volume element of the metric $\psi_1^*(g)$ on $\mathbb{C}_*^m$. For any $Z=(z_0=1,z_1,\dots,z_m)$ identified with the point $(z_k)_{1\leqslant k\leqslant m}$ of $\mathbb{C}_*^m$ (i.e. $x_k=|z_k|^2\neq 0$ for all k), if $I_h=\{d_0+\cdots+d_{h-1},\dots,d_0+\cdots+d_h-1\}$, with $d_0=0$, $h\in\{1,\dots,n\}$, we set

$$t_h=\sum_{j\in I_h}x_j,\qquad T=\sum_{h=1}^n\left(\sum_{j\in I_h}x_j^{\alpha_j}\right),\quad \text{where } \alpha_j=a_h \text{ if } j\in I_h,\quad \text{and}\quad K=\log\left(T^n\prod_{h=1}^n t_h^{d_h-a_h}\right).$$

Let us also put $J_1=\{1,\dots,d_1-1\}$ and $J_h=I_h$ for $h\geqslant 2$.

Proposition 1. (1) *Let $g_{\lambda\overline{\mu}}=\partial_{\lambda\overline{\mu}}K$ and $M=(g_{\lambda\overline{\mu}})_{1\leqslant\lambda,\mu\leqslant m}$. Then*

$$\det M=\prod_{h=1}^n\det B_h+\sum_{h=1}^n\det B_1\cdots\det B_{h-1}\Gamma_{(h)}\det B_{h+1}\cdots\det B_n,$$

where $\det B_h$ and $\Gamma_{(h)}$ are defined by $(A.3)$ and $(A.7)$ in the subsequent proof.

(2) *There is a constant C such that, at any point $Z=(z_k)_{1\leqslant k\leqslant m}$ satisfying $0<x_k\leqslant 1$ for all $k=1,\dots,m$, we have*

$$\det M\leqslant C\prod_{h=2}^n t_h^{a_h-d_h}.$$

Proof. To avoid to interrupt the main stream of the article, the proof is given in Appendix A. It is much more tricky than in the case $a_1=\cdots=a_n=1$ studied in [12]. The explicit value of $\det M$ given in part (1), in particular the cumbersome expressions of $\det B_h$ and $\Gamma_{(h)}$, is essential to get the upper bound of the volume element obtained in part (2). This upper bound is itself crucial in the evaluation of Tian's invariant $\alpha_G(X)$.

3. Proof of Theorem 2

3.1. Minoration of admissible functions

The functions φ we consider are g-admissible ($g_{\lambda\overline{\mu}}+\partial_{\lambda\overline{\mu}}\varphi>0$) and G-invariant. We use the notations of 2.2, in particular as regard the expressions $\varphi_h=\varphi\circ\psi_h$ of φ in parametrizations ψ_h of V. Recall that we write

$$\varphi_1(1,z_1,\dots,z_m)=\varphi(1,x_1,\dots,x_m)$$

since φ_1 depends only on the $x_k=|z_k|^2$. Thus,

$$\frac{\partial^2\varphi}{\partial z_j\partial\bar z_k}=\delta_{jk}\partial_j\varphi+\bar z_j z_k\partial_{jk}\varphi,$$

where $\partial_j=\partial/\partial x_j$ and $\partial_{jk}=\partial^2/\partial x_j\partial x_k$.

To get lower bounds of g-admissible, G-invariant functions, we proceed in a sequence of propositions. For $p\in X$, we express that the restriction of $K+\varphi$ to conveniently chosen holomorphic curves γ of X starting from p is subharmonic; the curves are such that informations concerning $K+\varphi$ at the extremity of γ can be obtained by virtue of G invariance properties of K and φ. In this way, we progressively reduce the number of variables and finally we obtain a logarithmic upper estimate of $-(K+\varphi)$.

Proposition 2. *Let $\varphi\in C^\infty(X)$ be a g-admissible and G-invariant function on $X=X_{[d],[a]}$. For $x_1,\dots,x_m>0$, if*

$$\gamma=\left(\prod_{k=1}^{d_1-1}x_k\right)^{1/(d_1-1)},$$

we have

$$-(K+\varphi)(1,x_1,\dots,x_{b_1},x_{b_1+1},\dots,x_m)\leqslant-(K+\varphi)\bigl(1,\gamma^{[d_1-1]},x_{b_1+1},\dots,x_m\bigr),$$

where

$$K=\log\left(\sum_{h=1}^{n}\left[\sum_{j\in I_h}x_j^{a_h}\right]\right)^n\prod_{h=1}^{n}\left(\sum_{j\in I_h}x_j\right)^{d_h-a_h}$$

and, for any $p\in\mathbb{N}$, $\gamma^{[p]}=(\gamma,\dots,\gamma)\in\mathbb{R}_*^p$.

Proof. First step. For $1\leqslant k\leqslant d_1-1$, $0<\zeta\leqslant 1$ and $x_{k+1},\dots,x_m>0$, let us prove the following inequality:

$$\begin{aligned}&-(K+\varphi)(1,\zeta,\dots,\zeta,x_{k+1},\dots,x_m)\\&\quad\leqslant-(K+\varphi)\left(1,\dots,1,\frac{x_{k+1}}{\zeta^{1/b}},\dots,\frac{x_{b_1}}{\zeta^{1/b}};\frac{x_{b_1+1}}{\zeta^{a_1/a_2b}},\dots,\frac{x_{b_2}}{\zeta^{a_1/a_2b}};\dots;\frac{x_{d_1+d_2+\cdots+d_n-1}}{\zeta^{a_1/a_nb}},\dots,\frac{x_m}{\zeta^{a_1/a_nb}}\right)\\&\qquad+a_1\left(\sum_{h=1}^{n}\frac{d_h}{a_h}\right)\log\frac{1}{\zeta^{1/b}},\end{aligned}\tag{2}$$

with $b=(k+1)/k$. For $s>0$, we set

$$\Phi(s)=s\frac{\mathrm{d}}{\mathrm{d}s}(K+\varphi)\big(\psi(s)\big),\tag{3}$$

where

$$\psi(s)\;=\;\big(1,s^b\zeta,\dots,s^b\zeta,sx_{k+1},\dots,sx_{b_1};s^{a_1/a_2}x_{b_1+1},\dots,s^{a_1/a_2}x_{b_2};\dots;s^{a_1/a_n}x_{b_{n-1}+1},\dots,s^{a_1/a_n}x_m\big).$$

In $\mathbb{C}_*^m=\{(1,\zeta_1,\dots,\zeta_m)\in\mathbb{C}^{m+1};\prod_{k=1}^m\zeta_k\neq 0\}$, let us consider the curve defined, for any complex number $\sigma\in\mathbb{C}-]-\infty,0]$, by

$$\begin{aligned}\gamma(\sigma)\;=\;&\big(1,\sigma^b\sqrt{\zeta},\dots,\sigma^b\sqrt{\zeta},\sigma\sqrt{x_{k+1}},\dots,\sigma\sqrt{x_{b_1}};\sigma^{a_1/a_2}\sqrt{x_{b_1+1}},\dots,\sigma^{a_1/a_2}\sqrt{x_{b_2}};\\&\sigma^{a_1/a_n}\sqrt{x_{b_{n-1}+1}},\dots,\sigma^{a_1/a_n}\sqrt{x_m}\big),\end{aligned}$$

where we take the principal determinations of the powers σ^b,$\sigma^{a_1/a_2},\dots,\sigma^{a_1/a_n}$. Since $K+\varphi$ is strictly plurisubharmonic, hence its Laplacian is positive. On the other hand, it depends only on $s=\sigma\overline{\sigma}$ and we write it $\psi=\psi(s)$, with ψ defined above. But,

$$\frac{\partial^2\psi}{\partial\sigma\partial\overline{\sigma}}(\sigma\overline{\sigma})=\psi'(\sigma\overline{\sigma})+\sigma\overline{\sigma}\psi''(\sigma\overline{\sigma})=\left(s\frac{\mathrm{d}\psi}{\mathrm{d}s}(s)\right)'\geqslant 0.$$

Thus, $\Phi'(s)\geqslant 0$ and Φ is an increasing function, with $\Phi(s)$ explicitly given by

$$\Phi(s)=bs^b\zeta\big(\partial_1(K+\varphi)+\cdots+\partial_k(K+\varphi)\big)+\sum_{p=k+1}^{d_1-1}sx_p\partial_p(K+\varphi)+\sum_{h=2}^{n}\sum_{p\in I_h}\frac{a_1}{a_h}s^{a_1/a_h}x_p\partial_p(K+\varphi),$$

where the derivatives are taken at $\psi(s)$.

Setting $s_0=\zeta^{-1/b}\geqslant 1$, we compute all derivatives of $K+\varphi$ at

$$P_0=\psi(s_0)=\big(1^{[k+1]},y_{k+1},\dots,y_m\big)$$

with $y_p=s_0x_p$ for $p=k+1,\dots,d_1-1$ and $y_p=s_0^{a_1/a_h}x_p$ if $p\in I_h$ and $h\geqslant 2$.

Consequently, for $1\leqslant s\leqslant s_0$,

$$\Phi(1)\;\leqslant\;\Phi(s)\leqslant\Phi(s_0)=b\sum_{p=1}^{k}\partial_p(K+\varphi)+\sum_{p=k+1}^{d_1-1}y_p\partial_p(K+\varphi)+\sum_{h=2}^{n}\frac{a_1}{a_h}\left[\sum_{p\in I_h}\partial_p(K+\varphi)\right].\tag{4}$$

Let us evaluate $\Phi(s_0)$.

By definition of K, we write

$$\left[b\sum_{p=1}^{k}\partial_p K+\sum_{p=k+1}^{d_1-1}y_p\partial_p K+\sum_{h=2}^{n}\frac{a_1}{a_h}\left(\sum_{p\in I_h}y_p\partial_p K\right)\right](P_0)$$
$$=b\sum_{p=1}^{k}\frac{na_1}{T}+\sum_{p=k+1}^{d_1-1}\frac{na_1y_p^{a_1-1}y_p}{T}+\sum_{h=2}^{n}\sum_{p\in I_h}\frac{a_1}{a_h}\frac{na_hy_p^{a_h-1}y_p}{T}+b\sum_{p=1}^{k}\frac{d_1-a_1}{t_1}+\sum_{p=k+1}^{d_1-1}\frac{(d_1-a_1)y_p}{t_1}$$
$$+\sum_{h=2}^{n}\frac{a_1}{a_h}\sum_{p\in I_h}\frac{(d_h-a_h)y_p}{t_h},$$

where

$$T=k+1+\sum_{p=k+1}^{d_1-1}y_p^{a_1}+\sum_{h=2}^{n}\sum_{p\in I_h}y_p^{a_h},$$

$$t_1=k+1+\sum_{p=k+1}^{d_1-1}y_p\quad\text{and}\quad t_h=\sum_{p\in I_h}y_p\quad\text{when } 2\leqslant h\leqslant n.$$

Hence,

$$\left[b\sum_{p=1}^{k}\partial_p K+\sum_{p=k+1}^{d_1-1}y_p\partial_p K+\sum_{h=2}^{n}\frac{a_1}{a_h}\left(\sum_{p\in I_h}y_p\partial_p K\right)\right](P_0)$$
$$=\frac{na_1}{T}\left[k+1+\sum_{p=k+1}^{d_1-1}y_p^{a_1}+\sum_{h=2}^{n}\left(\sum_{p\in I_h}y_p^{a_h}\right)\right]+\frac{d_1-a_1}{t_1}\left(k+1+\sum_{p=k+1}^{d_1-1}y_p\right)+\sum_{h=2}^{n}\frac{a_1}{a_h}\frac{d_h-a_h}{t_h}\left(\sum_{p\in I_h}y_p\right)$$
$$=na_1+\sum_{h=1}^{n}\frac{a_1}{a_h}(d_h-a_h)=a_1\sum_{h=1}^{n}\frac{d_h}{a_h}.\tag{5}$$

Let us now show that

$$\left[b\sum_{p=1}^{k}\partial_p\varphi+\sum_{p=k+1}^{d_1-1}y_p\partial_p\varphi+\sum_{h=2}^{n}\frac{a_1}{a_h}\left(\sum_{p\in I_h}y_p\partial_p\varphi\right)\right](P_0)=0.\tag{6}$$

We shall use the invariance of φ with respect to the automorphisms $\sigma_{0,1},\dots,\sigma_{0,k}$. As we saw in 2.2, if $1\leqslant i\leqslant d_1-1$, since φ is $\sigma_{0,i}$-invariant, we have

$$\varphi(1,u_1,\dots,u_m)=\varphi\left(1,\frac{u_1}{u_i},\dots,\frac{u_{j-1}}{u_i},\frac{1}{u_i},\frac{u_{j+1}}{u_i},\dots,\frac{u_{b_1}}{u_i};\frac{u_{b_1+1}}{u_i^{a_1/a_2}},\dots,\frac{u_{b_2}}{u_i^{a_1/a_2}};\dots;\frac{u_{b_{n-1}+1}}{u_i^{a_1/a_n}},\dots,\frac{u_m}{u_i^{a_1/a_n}}\right),\tag{7}$$

for any $u_1,\dots,u_m>0$. Thus, for any $\eta>0$ and any $i\in\{1,\dots,k\}$, the following relation is satisfied:

$$\varphi\left(1,\eta^{[k]},y_{k+1},\dots,y_m\right)=\left(1^{[i]},\frac{1}{\eta},1^{[k-i]},\frac{y_{k+1}}{\eta},\dots,\frac{y_{b_1}}{\eta};\frac{y_{b_1+1}}{\eta^{a_1/a_2}},\dots,\frac{y_{b_2}}{\eta^{a_1/a_2}};\dots;\frac{y_{b_{n-1}+1}}{\eta^{a_1/a_n}},\dots,\frac{y_m}{\eta^{a_1/a_n}}\right).$$

Let us differentiate this equality with respect to η at $\eta=1$. We obtain at point P_0:

$$\sum_{j=1}^{k}\partial_j\varphi=-\partial_i\varphi-\sum_{p=k+1}^{d_1-1}y_p\partial_p\varphi-\sum_{h=2}^{n}\sum_{p\in I_h}\frac{a_1}{a_h}y_p\partial_p\varphi.$$

By summation on $i=1,\dots,k$, we deduce that

$$(k+1)\sum_{j=1}^{k}\partial_j\varphi+k\sum_{p=k+1}^{d_1-1}y_p\partial_p\varphi+k\sum_{h=2}^{n}\sum_{p\in I_h}\frac{a_1}{a_h}y_p\partial_p\varphi=0,$$

which is equality (6) since $b=(k+1)/k$.

Combining (5), (6) and the value of $\Phi(s_0)$ as given in (4), we see that $\Phi(s_0)=a_1\sum_{h=1}^{n}\frac{d_h}{a_h}$. Then, taking into account (4) and the Definition (3) of $\Phi(s)$, we obtain for $1\leqslant s\leqslant s_0$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\big[(K+\varphi)\big(\psi(s)\big)\big] \leqslant a_1 \sum_{h=1}^{n} \frac{d_h}{a_h}. \tag{8}$$

Finally, integrating this inequality between 1 and $s_0 = \zeta^{-b}$ yields (2).

Second step. Let $1 \leqslant k \leqslant d_1 - 1$. We want to prove by induction on k that, for $x_1, \dots, x_k > 0$ and $\gamma_k = (\prod_{j=1}^{k} x_j)^{1/k}$, we have

$$-(K+\varphi)(1, x_1, \dots, x_k, x_{k+1}, \dots, x_m) \leqslant -(K+\varphi)(1, \gamma_k, \dots, \gamma_k, x_{k+1}, \dots, x_m). \tag{L_k}$$

So let us assume (L_{k-1}) is valid for some $k \in \{2, \dots, d_1 - 1\}$ (assertion clearly true when $k = 2$); we have to show (L_k). First, the definition of K yields

$$\begin{aligned}
&K\left(1, \frac{x_1}{x_k}, \dots, \frac{x_{k-1}}{x_k}, \frac{1}{x_k}, \frac{x_{k+1}}{x_k}, \dots, \frac{x_{b_1}}{x_k}; \frac{x_{b_1+1}}{x_k^{a_1/a_2}}, \dots, \frac{x_{b_2}}{x_k^{a_1/a_2}}; \dots; \frac{x_{b_{n-1}+1}}{x_k^{a_1/a_n}}, \dots, \frac{x_m}{x_k^{a_1/a_n}}\right) \\
&\quad = \sum_{h=1}^{n} (d_h - a_h) \log\left(x_k^{-a_1/a_h} \sum_{j \in I_h} x_j\right) + n \log\left(x_k^{-a_1} \sum_{h=1}^{n} \sum_{j \in I_h} x_j^{a_h}\right) \\
&\quad = K(1, x_1, \dots, x_m) + a_1 \left(\sum_{h=1}^{n} \frac{d_h - a_h}{a_h} + n\right) \log \frac{1}{x_k} \\
&\quad = K(1, x_1, \dots, x_m) + a_1 \sum_{h=1}^{n} \frac{d_h}{a_h} \log \frac{1}{x_k},
\end{aligned}$$

which implies

$$\begin{aligned}
&K\big(1, \gamma_{k-1}^{[k-1]}, x_k, \dots, x_m\big) \\
&\quad = K\left(1, \left(\frac{\gamma_{k-1}}{x_k}\right)^{[k-1]}, \frac{1}{x_k}, \frac{x_{k+1}}{x_k}, \dots, \frac{x_{b_1}}{x_k}; \frac{x_{b_1+1}}{x_k^{a_1/a_2}}, \dots, \frac{x_{b_2}}{x_k^{a_1/a_2}}; \dots; \frac{x_{b_{n-1}+1}}{x_k^{a_1/a_n}}, \dots, \frac{x_m}{x_k^{a_1/a_n}}\right) \\
&\qquad - a_1 \sum_{h=1}^{n} \frac{d_h}{a_h} \log \frac{1}{x_k}.
\end{aligned} \tag{9}$$

On the other hand, by $\sigma_{0,k}$-invariance of φ, we have (see 2.2):

$$\begin{aligned}
&\varphi\big(1, \gamma_{k-1}^{[k-1]}, x_k, \dots, x_m\big) \\
&\quad = \varphi\left(1, \left(\frac{\gamma_{k-1}}{x_k}\right)^{[k-1]}, \frac{1}{x_k}, \frac{x_{k+1}}{x_k}, \dots, \frac{x_{b_1}}{x_k}; \frac{x_{b_1+1}}{x_k^{a_1/a_2}}, \dots, \frac{x_{b_2}}{x_k^{a_1/a_2}}; \dots; \frac{x_{b_{n-1}+1}}{x_k^{a_1/a_n}}, \dots, \frac{x_m}{x_k^{a_1/a_n}}\right).
\end{aligned} \tag{10}$$

Now taking into account the $\sigma_{i,j}$-invariance of φ for $1 \leqslant i < j \leqslant k$, we may assume that $x_1 \leqslant \cdots \leqslant x_k$; thus,

$$\gamma_{k-1} = \left(\prod_{j=1}^{k-1} x_j\right)^{1/(k-1)} \leqslant x_k \quad \text{and} \quad \zeta = \frac{\gamma_{k-1}}{x_k} \leqslant 1.$$

Hence, by (L_{k-1}), (9), (10) and (2) (written for $k-1$ instead of k), we obtain

$$\begin{aligned}
&-(K+\varphi)(1, x_1, \dots, x_m) \\
&\quad \leqslant -(K+\varphi)\big(1, \gamma_{k-1}^{[k-1]}, x_k, \dots, x_m\big) \\
&\quad \leqslant -(K+\varphi)\left(1, \zeta^{[k-1]}, \frac{1}{x_k}, \frac{x_{k+1}}{x_k}, \dots, \frac{x_{b_1}}{x_k}; \frac{x_{b_1+1}}{x_k^{a_1/a_2}}, \dots, \frac{x_{b_2}}{x_k^{a_1/a_2}}; \dots; \frac{x_{b_{n-1}+1}}{x_k^{a_1/a_n}}, \dots, \frac{x_m}{x_k^{a_1/a_n}}\right) + a_1 \sum_{h=1}^{n} \frac{d_h}{a_h} \log \frac{1}{x_k} \\
&\quad \leqslant -(K+\varphi)\left(1^{[k]}, \frac{1}{\zeta^c x_k}, \frac{x_{k+1}}{\zeta^c x_k}, \dots, \frac{x_{b_1}}{\zeta^c x_k}; \frac{x_{b_1+1}}{\zeta^{ca_1/a_2} x_k^{a_1/a_2}}, \dots, \frac{x_{b_2}}{\zeta^{ca_1/a_2} x_k^{a_1/a_2}}; \dots; \right. \\
&\qquad\qquad \left. \frac{x_{b_{n-1}+1}}{\zeta^{ca_1/a_n} x_k^{a_1/a_n}}, \dots, \frac{x_m}{\zeta^{ca_1/a_n} x_k^{a_1/a_n}}\right)
\end{aligned}$$

$$+a_1\sum_{h=1}^{n}\frac{d_h}{a_h}\log\frac{1}{\zeta^c}+a_1\sum_{h=1}^{n}\frac{d_h}{a_h}\log\frac{1}{x_k},\quad \text{with } c=\frac{k-1}{k}.$$

Consequently, since

$$\zeta^c x_k=\left(\frac{\gamma_{k-1}}{x_k}\right)^{(k-1)/k}x_k=\left(\prod_{j=1}^{k-1}x_j\right)^{1/k}x_k^{1/k}=\gamma_k$$

and

$$\zeta^{ca_1/a_h}x_k^{a_1/a_h}=\left(\zeta^c x_k\right)^{a_1/a_h}=\gamma_k^{a_1/a_h},\quad \text{for } h\geqslant 2 \text{ and } k\in I_h,$$

we get:

$$\begin{aligned}
&-(K+\varphi)(1,x_1,\ldots,x_m)\\
&\quad\leqslant -(K+\varphi)\left(1^{[k]},\frac{1}{\gamma_k},\frac{x_{k+1}}{\gamma_k},\ldots,\frac{x_{b_1}}{\gamma_k};\frac{x_{b_1+1}}{\gamma_k^{a_1/a_2}},\ldots,\frac{x_{b_2}}{\gamma_k^{a_1/a_2}};\ldots;\frac{x_{b_{n-1}+1}}{\gamma_k^{a_1/a_n}},\ldots,\frac{x_m}{\gamma_k^{a_1/a_n}}\right)+\sum_{h=1}^{n}\frac{d_h}{a_h}\log\frac{1}{\gamma_k}\\
&\quad=-(K+\varphi)\left(1,\gamma_k^{[k]},x_{k+1},\ldots,x_m\right)\\
&\qquad+\left[K\left(1,\gamma_k^{[k]},x_{k+1},\ldots,x_m\right)-K\left(1^{[k]},\frac{1}{\gamma_k},\frac{x_{k+1}}{\gamma_k},\ldots,\frac{x_{b_1}}{\gamma_k};\frac{x_{b_1+1}}{\gamma_k^{a_1/a_2}},\ldots,\frac{x_{b_2}}{\gamma_k^{a_1/a_2}};\ldots;\frac{x_{b_{n-1}+1}}{\gamma_k^{a_1/a_n}},\ldots,\frac{x_m}{\gamma_k^{a_1/a_n}}\right)\right]\\
&\qquad+a_1\sum_{h=1}^{n}\frac{d_h}{a_h}\log\frac{1}{\gamma_k}\\
&\qquad\text{(by } \sigma_{0,k}\text{-invariance of } \varphi)\\
&\quad=-(K+\varphi)\left(1,\gamma_k^{[k]},x_{k+1},\ldots,x_m\right)
\end{aligned}$$

(because the bracket involving the difference of the values taken by K in two points is equal to $(-a_1\sum_{h=1}^{n}\frac{d_h}{a_h}\log\frac{1}{\gamma_k})$).

Finally, tracing through the inequalities, we see that we have obtained (I_k) as required.

Proposition 3. *Let $\varphi\in C^\infty(X)$ be a g-admissible and G-invariant function. For $x_1,\ldots,x_m>0$, we set*

$$\zeta_1=\left(\prod_{j=1}^{d_1-1}x_j\right)^{1/(d_1-1)}\quad \textit{and}\quad \zeta_h=\left(\prod_{j=b_{h-1}+1}^{d_1+\cdots+d_h-1}x_j\right)^{1/d_h},\quad \textit{if } 2\leqslant h\leqslant n.$$

When $\zeta\in\mathbb{R}$, $\zeta^{[d]}$ denotes the vector $(\zeta,\ldots,\zeta)\in\mathbb{R}^d$. Then, the following inequality is satified:

$$-(K+\varphi)(1,x_1,\ldots,x_m)\leqslant -(K+\varphi)\left(1,\zeta_1^{[d_1-1]},\zeta_2^{[d_2]},\ldots,\zeta_n^{[d_n]}\right).$$

Proof. Let $j\in\{1,\ldots,n\}$ and $h\in\{2,\ldots,n\}$. We suppose the inequality

$$(*)_j\quad -(K+\varphi)(1,x_1,\ldots,x_m)\leqslant -(K+\varphi)\left(1,\zeta_1^{[d_1-1]},\zeta_2^{[d_2]},\ldots,\zeta_j^{[d_j]},x_{d_1+\cdots+d_j},\ldots,x_m\right)$$

valid when $j=h-1$ and we prove $(*)_h$. Since $(*)_1$ is true according to the previous proposition, $(*)_n$ will thus be obtained by induction. We have to show that

$$-(K+\varphi)\left(1,\zeta_1^{[d_1-1]},\zeta_2^{[d_2]},\ldots,\zeta_{h-1}^{[d_{h-1}]},x_{b_{h-1}+1},\ldots,x_m\right)\leqslant -(K+\varphi)\left(1,\zeta_1^{[d_1-1]},\zeta_2^{[d_2]},\ldots,\zeta_h^{[d_h]},x_{b_h+1},\ldots,x_m\right).$$

We set $c=b_{h-1}+1$. Using the change of parametrization $\psi_h^{-1}\circ\psi_1$ of V (see 2.1(b)), the definition of K and Proposition 2 written in parametrization ψ_h of V yields:

$$\begin{aligned}
&-(K+\varphi)\left(1,\zeta_1^{[d_1-1]},\zeta_2^{[d_2]},\ldots,\zeta_h^{[d_h]},x_c,\ldots,x_m\right)\\
&\quad=-(K+\varphi)\left(\frac{1}{x_c^{a_h/a_1}},\frac{\zeta_1^{[d_1-1]}}{x_c^{a_h/a_1}};\frac{\zeta_2^{[d_2]}}{x_c^{a_h/a_2}};\ldots;\frac{\zeta_{h-1}^{[d_{h-1}]}}{x_c^{a_h/a_{h-1}}};1,\frac{x_{c+1}}{x_c},\ldots,\frac{x_{b_h}}{x_c};\frac{x_{b_h+1}}{x_c^{a_h/a_{h+1}}},\ldots,\frac{x_m}{x_c^{a_h/a_n}}\right)
\end{aligned}$$

$$+\left[K\left(\frac{1}{x_c^{a_h/a_1}},\frac{\zeta_1^{[d_1-1]}}{x_c^{a_h/a_1}};\frac{\zeta_2^{[d_2]}}{x_c^{a_h/a_2}};\dots;\frac{x_m}{x_c^{a_h/a_n}}\right)-K\big(1,\zeta_1^{[d_1-1]},\zeta_2^{[d_2]},\dots,\zeta_h^{[d_h]},x_c,\dots,x_m\big)\right]$$
$$\leqslant -(K+\varphi)\left(\frac{1}{x_c^{a_h/a_1}},\frac{\zeta_1^{[d_1-1]}}{x_c^{a_h/a_1}};\frac{\zeta_2^{[d_2]}}{x_c^{a_h/a_2}};\dots;\frac{\zeta_{h-1}^{[d_{h-1}]}}{x_c^{a_h/a_{h-1}}};1,\rho^{[d_h-1]};\frac{x_{b_h+1}}{x_c^{a_h/a_{h+1}}},\dots,\frac{x_m}{x_c^{a_h/a_n}}\right)+a_h\sum_{j=1}^n\frac{d_j}{a_j}\log\frac{1}{x_c},$$

with $\rho=\frac{1}{x_c}(\prod_{p=c+1}^{c+d_h-1}x_p)^{1/d_h-1}$. Now, we invoke the first step of the proof of Proposition 2(2) to get an upper bound of the term in $-(K+\varphi)$ at the right-hand side of the last inequality. Setting $\alpha=(d_h-1)/d_h$, we obtain:

$$-(K+\varphi)\big(1,\zeta_1^{[d_1-1]},\zeta_2^{[d_2]},\dots,\zeta_{h-1}^{[d_{h-1}]},x_c,\dots,x_m\big)$$
$$\leqslant -(K+\varphi)\left(\frac{1}{\rho^{\alpha a_h/a_1}x_c^{a_h/a_1}},\frac{\zeta_1^{[d_1-1]}}{\rho^{\alpha a_h/a_1}x_c^{a_h/a_1}};\frac{\zeta_2^{[d_2]}}{\rho^{\alpha a_h/a_2}x_c^{a_h/a_2}};\dots;\frac{\zeta_{h-1}^{[d_{h-1}]}}{\rho^{\alpha a_h/a_{h-1}}x_c^{a_h/a_{h-1}}};1^{[d_h]};\right.$$
$$\left.\frac{x_{b_h+1}}{\rho^{\alpha a_h/a_{h+1}}x_c^{a_h/a_{h+1}}},\dots,\frac{x_m}{\rho^{\alpha a_h/a_n}x_c^{a_h/a_n}}\right)+a_h\sum_{j=1}^n\frac{d_j}{a_j}\left(\log\frac{1}{\rho^\alpha}+\log\frac{1}{x_c}\right)$$
$$=-(K+\varphi)\left(\frac{1}{\zeta_h^{a_h/a_1}},\frac{\zeta_1^{[d_1-1]}}{\zeta_h^{a_h/a_1}};\frac{\zeta_2^{[d_2]}}{\zeta_h^{a_h/a_2}};\dots;\frac{\zeta_{h-1}^{[d_{h-1}]}}{\zeta_h^{a_h/a_{h-1}}};1^{[d_h]};\frac{x_{b_h+1}}{\zeta_h^{a_h/a_{h+1}}},\dots,\frac{x_m}{\zeta_h^{a_h/a_n}}\right)+a_h\sum_{j=1}^n\frac{d_j}{a_j}\log\frac{1}{\zeta_h},$$

$$\text{since}\quad \rho^\alpha x_c=\frac{1}{x_c^{(d_h-1)/d_h}}\left(\prod_{j=c+1}^{c+d_h-1}x_j\right)^{1/d_h}x_c=\left(\prod_{j=c}^{c+d_h-1}x_j\right)^{1/d_h}=\zeta_h.$$

Using once more the definition of K and the change $\psi_1^{-1}\circ\psi_h$ of parametrization of V, we remark that the term in $-(K+\varphi)$ at the end of the last equality is equal to

$$-(K+\varphi)\big(1,\zeta_1^{[d_1-1]};\zeta_2^{[d_2]};\dots;\zeta_h^{[d_h]};x_{b_h+1},\dots,x_m\big)-a_h\sum_{j=1}^n\frac{d_j}{a_j}\log\frac{1}{\zeta_h}.$$

Thus we conclude that

$$-(K+\varphi)(x_1,\dots,x_m)\leqslant -(K+\varphi)\big(1,\zeta_1^{[d_1-1]};\zeta_2^{[d_2]};\dots;\zeta_h^{[d_h]};x_{b_h+1},\dots,x_m\big),$$

which proves $(*)_h$.

Proposition 4. *Under the hypothesis $a_1=\dots=a_n=a$ and $d_1=\dots=d_n=d$, let $\varphi\in C^\infty(X_{[d],[a]})$ be a g-admissible, G-invariant function. Then, for $x_1,\dots,x_m>0$, we have*

$$-(K+\varphi)(1,x_1,\dots,x_m)\leqslant -(K+\varphi)(1,\dots,1)-\log x_1\cdots x_m.$$

Proof. (a) First, let us prove that for $\zeta_1,\dots,\zeta_n>0$ and

$$\eta=\left(\prod_{j=2}^n\zeta_j\right)^{1/(n-1)},$$

we have

$$-(K+\varphi)\big(1,\zeta_1^{[d-1]},\zeta_2^{[d]},\dots,\zeta_n^{[d]}\big)\leqslant -(K+\varphi)\big(1,\zeta_1^{[d-1]},\eta,\dots,\eta\big),\tag{11}$$

where $\zeta^{[d]}=(\zeta,\dots,\zeta)\in\mathbb{R}^d$.

The G-invariance of φ allows us to suppose that $\zeta_2\geqslant\dots\geqslant\zeta_n$. To prove (11), we show that, if $2\leqslant h\leqslant n-1$ and $t\geqslant\zeta_{h+1}\geqslant\dots\geqslant\zeta_n$, then, for $\lambda=t^{(h-1)/h}(\zeta_{h+1})^{1/h}$, the following inequality holds:

$$\begin{aligned}T_h&=-(K+\varphi)\big(1,\zeta_1^{[d-1]},t^{[d]},\dots,t^{[d]},\zeta_{h+1}^{[d]},\dots,\zeta_n^{[d]}\big)\\&\leqslant -(K+\varphi)\big(1,\zeta_1^{[d-1]},\lambda^{[d]},\dots,\lambda^{[d]},\zeta_{h+2}^{[d]},\dots,\zeta_n^{[d]}\big).\end{aligned}\tag{12}$$

(b) *Proof of* (12). As in the first step of the proof of Proposition 2 one proves that the function

$$\psi(s)=s\frac{\mathrm{d}}{\mathrm{d}s}(K+\varphi)\big(1^{[(h-1)d]},\big(s^h\zeta\big)^{[d]},sx_{hd},\dots,sx_m\big),$$

where

$$\zeta=\zeta_{h+1}/t,\qquad x_j=\frac{\zeta_{k+1}}{t}\quad\text{if } j\in I_k \text{ and } k=h+1,\dots,n-1,$$
$$x_{(n-1)d}=\frac{1}{t}\quad\text{and}\quad x_j=\frac{\zeta_1}{t}\quad\text{if } (n-1)d+1\leqslant j\leqslant nd-1,$$

is increasing. The derivatives of $(K+\varphi)$ being taken at point

$$P=\big(1^{[(h-1)d]},\big(s^h\zeta\big)^{[d]},sx_{hd},\dots,sx_m\big),$$

we have

$$\psi(s)=hs^h\zeta\sum_{j=(h-1)d}^{hd-1}\partial_j(K+\varphi)+s\sum_{j=hd}^{m}x_j\partial_j(K+\varphi).$$

So, for $1\leqslant s\leqslant s_0=\zeta^{-1/h}$, we infer

$$\frac{\mathrm{d}}{\mathrm{d}s}(K+\varphi)\big(1^{[(h-1)d]},\big(s^h\zeta\big)^{[d]},sx_{hd},\dots,sx_m\big)\leqslant\frac{\psi(s_0)}{s}.\tag{13}$$

Let us compute $\psi(s_0)$. Defining

$$P_0=\big(1^{[(h-1)d]},\big(s_0^h\zeta\big)^{[d]},s_0x_{hd},\dots,s_0x_m\big)=\big(1^{[hd]},s_0x_{hd},\dots,s_0x_m\big),$$

we have

$$\begin{aligned}\psi(s_0)&=\left\{hs_0^h\zeta\sum_{j=(h-1)d}^{hd-1}\partial_j(K+\varphi)+s_0\sum_{j=hd}^{m}x_j\partial_j(K+\varphi)\right\}(P_0)\\&=\left\{h\sum_{j=(h-1)d}^{hd-1}\partial_j(K+\varphi)+\sum_{j=hd}^{m}s_0x_j\partial_j(K+\varphi)\right\}(P_0).\end{aligned}$$

Then, by definition of K, we get

$$\begin{aligned}&\left(h\sum_{j=(h-1)d}^{hd-1}\partial_jK+\sum_{j=hd}^{m}s_0x_j\partial_jK\right)(P_0)\\&\quad=\sum_{j=(h-1)d}^{hd-1}\frac{nha}{hd+s_0^ax_{hd}^a+\cdots+s_0^ax_m^a}+\sum_{j=(h-1)d}^{hd-1}\frac{(d-a)h}{d}+\sum_{j=hd}^{m}\frac{nas_0^{a-1}x_j^{a-1}s_0x_j}{hd+s_0^ax_{hd}^a+\cdots+s_0^ax_m^a}+(n-h)(d-a)\\&\quad=na+(d-a)h+(n-h)(d-a)=m+1.\end{aligned}\tag{14}$$

On the other hand, let us show that

$$\left(h\sum_{j=(h-1)d}^{hd-1}\partial_j\varphi+\sum_{j=hd}^{m}s_0x_j\partial_j\varphi\right)(P_0)=0.\tag{15}$$

Since φ is G-invariant, we have, at point P_0:

$$\sum_{j\in I_2}\partial_j\varphi=\cdots=\sum_{j\in I_h}\partial_j\varphi$$

(let us remind that $I_h=\{(h-1)d\leqslant j\leqslant hd-1\}$ if $h\geqslant 2$) and, for $u>0$,

$$\varphi\big(1^{[(h-1)d]},u^{[d]},s_0x_{hd},\dots,s_0x_m\big)=\varphi\left(1^{[d]},\left(\frac{1}{u}\right)^{[(h-1)d]},\frac{s_0x_{hd}}{u},\dots,\frac{s_0x_m}{u}\right).$$

Differentiating the previous equality with respect to u at $u=1$ gives, always at P_0,

$$\sum_{j\in I_h} \partial_j \varphi = -\sum_{j\in I_2} \partial_j \varphi - \cdots - \sum_{j\in I_h} \partial_j \varphi - \sum_{j=hd}^{m} s_0 x_j \partial_j \varphi,$$

which implies

$$h \sum_{j=(h-1)d}^{hd-1} \partial_j \varphi + \sum_{j=hd}^{m} s_0 x_j \partial_j \varphi = 0$$

and finally yields relation (15).

Combining (14) and (15), we have $\psi(s_0) = m+1$ and, by virtue of (13), we see that

$$\frac{\mathrm{d}}{\mathrm{d}s}(K+\varphi)\big(1^{[(h-1)d]}, \big(s^h \zeta\big)^{[d]}, s x_{hd}, \ldots, s x_m\big) \leqslant \frac{m+1}{s}.$$

Let us integrate the previous inequality between 1 and $s_0 = \zeta^{-1/h}$. Taking into account the values, given at the beginning of the proof, of λ, ζ, and x_k, for $hd \leqslant k \leqslant m$, we obtain:

$$\begin{aligned}
&(K+\varphi)\left(1^{[hd]}, \left(\frac{\zeta_{h+2}}{\lambda}\right)^{[d]}, \ldots, \left(\frac{\zeta_n}{\lambda}\right)^{[d]}, \frac{1}{\lambda}, \left(\frac{\zeta_1}{\lambda}\right)^{[d-1]}\right) \\
&\quad - (K+\varphi)\left(1^{[(h-1)d]}, \left(\frac{\zeta_{h+1}}{t}\right)^{[d]}, \left(\frac{\zeta_{h+2}}{t}\right)^{[d]}, \ldots, \left(\frac{\zeta_n}{t}\right)^{[d]}, \frac{1}{t}, \left(\frac{\zeta_1}{t}\right)^{[d-1]}\right) \\
&\quad \leqslant (m+1)\log\left(\frac{\zeta_{h+1}}{t}\right)^{-1/h}.
\end{aligned} \tag{16}$$

The G-invariance of φ and the definition of K imply

$$\begin{aligned}
&-(K+\varphi)\big(1, \zeta_1^{[d-1]}, t^{[d]}, \ldots, t^{[d]}, \zeta_{h+1}^{[d]}, \ldots, \zeta_n^{[d]}\big) \\
&\quad = -(K+\varphi)\left(1^{[(h-1)d]}, \frac{\zeta_{h+1}^{[d]}}{t}, \ldots, \frac{\zeta_n^{[d]}}{t}, \frac{1}{t}, \frac{\zeta_1^{[d-1]}}{t}\right) + (m+1)\log\frac{1}{t}
\end{aligned}$$

and

$$\begin{aligned}
&-(K+\varphi)\big(1, \zeta_1^{[d-1]}, \lambda^{[d]}, \ldots, \lambda^{[d]}, \zeta_{h+2}^{[d]}, \ldots, \zeta_n^{[d]}\big) \\
&\quad = -(K+\varphi)\left(1^{[hd]}, \frac{\zeta_{h+2}^{[d]}}{\lambda}, \ldots, \frac{\zeta_n^{[d]}}{\lambda}, \frac{1}{\lambda}, \frac{\zeta_1^{[d-1]}}{\lambda}\right) + (m+1)\log\frac{1}{\lambda}.
\end{aligned}$$

Inserting these two equalities in (16) gives:

$$\begin{aligned}
&-(K+\varphi)\big(1, \zeta_1^{[d-1]}, t^{[d]}, \ldots, t^{[d]}, \zeta_{h+1}^{[d]}, \ldots, \zeta_n^{[d]}\big) \\
&\quad \leqslant -(K+\varphi)\big(1, \zeta_1^{[d-1]}, \lambda^{[d]}, \ldots, \lambda^{[d]}, \zeta_{h+2}^{[d]}, \ldots, \zeta_n^{[d]}\big) + (m+1)\log\left\{\frac{\lambda}{t}\left(\frac{\zeta_{h+1}}{t}\right)^{-1/h}\right\}.
\end{aligned}$$

Hence, since

$$\frac{\lambda}{t}\left(\frac{\zeta_{h+1}}{t}\right)^{-1/h} = 1,$$

we deduce (12) and thus (11).

(c) Given a G-invariant admissible function $\varphi \in C^\infty(X)$ and $x_1, \ldots, x_m > 0$, combining Proposition 3 and inequality (11) leads to

$$-(K+\varphi)(1, x_1, \ldots, x_m) \leqslant -(K+\varphi)\big(1, \zeta^{[d-1]}, \eta^{[m+1-d]}\big), \tag{17}$$

where

$$\zeta = (x_1 \cdots x_{d-1})^{1/(d-1)} \quad \text{and} \quad \eta = (x_d \cdots x_m)^{1/(m+1-d)}.$$

Hence, to conclude the proof of Proposition 4, we have to bound from above

$$-(K+\varphi)\big(1, \zeta^{[d-1]}, \eta^{[m+1-d]}\big)$$

by $-(K+\varphi)(1,\ldots,1) - \log x_1 \cdots x_m$. A slight modification of the proof of Lemma 6 in [12] gives this result.

The proof of Theorem 2 also needs the two following Lemmas 1 and 2.

Lemma 1. *There exists a constant C such that for any $\varphi \in C^\infty(X)$, g-admissible and G-invariant, we have*

$$-(K+\varphi)(1,\ldots,1) \leqslant C.$$

Proof. It is analogous to the proof given in [12, pp. 682–683], to which we refer.

Now, we establish a result which expresses the integral over X of any G-invariant function as the sum of n integrals over the unit polydisc D of $\mathbb{C}_*^m$.

Lemma 2. *Let φ be a G-invariant integrable function on $X = X_{[d],[a]}$. For any $h = 1,\ldots,n$, we denote by*

$$\varphi_h = \varphi \circ \psi_h = \varphi(z_0,\ldots,z_{b_{h-1}}, 1, z_{b_{h-1}+2},\ldots,z_m)$$

the local expression of φ in parametrisation ψ_h of V (see 2.1(b)). Then, if dv_h is the volume element of the metric $\psi_h^*(g)$ on $\mathbb{C}_*^m$, and if $a'_h = \prod_{j\neq h} a_j$, we have

$$\int_X \varphi \,\mathrm{d}v = \sum_{h=1}^n \frac{d_h}{a'_h} \int_D \varphi_h \,\mathrm{d}v_h.$$

Remark. If we impose that the argument of the first component of $Z_j = (z_k)_{k\in I_j}$ belong to some interval of length $2\pi/a_j$ for any $j \in \{1,\ldots,n\} - \{h\}$, we get a subset D_h of D and

$$\int_X \varphi \,\mathrm{d}v = \sum_{h=1}^n d_h \int_{D_h} \varphi_h \,\mathrm{d}v_h.$$

Proof. If $1 \leqslant h \leqslant n$, we set $Z_h = (z_k)_{k\in I_h}$ and $X_h = (x_k = |z_k|^2)_{k\in I_h}$. The notation $\widetilde{Z}_h$ means that the first component $z_{b_{h-1}+1}$ of Z_h is equal to one; on the other hand, $X_h \leqslant 1$ if $0 < x_k \leqslant 1$ for any $k \in I_h$. Since $\varphi_h(Z_1,\ldots,\widetilde{Z}_h,\ldots,Z_n)$ is invariant by multiplication of the z_k (of index $k \neq b_{h-1}+1$) by any $\mathrm{e}^{\mathrm{i}\theta}$, φ_h depends only on $X_1,\ldots,\widetilde{X}_h,\ldots,X_n$ and we write

$$\varphi_h = \varphi\big(Z_1,\ldots,\widetilde{Z}_h,\ldots,Z_n\big) = \varphi\big(X_1,\ldots,\widetilde{X}_h,\ldots,X_n\big).$$

We want to prove by induction on h that

$$\int_X \varphi \,\mathrm{d}v = \sum_{l=1}^h \frac{d_l}{a'_l} \int_{X_1,\ldots,\widetilde{X}_l,\ldots,X_h \leqslant 1} \varphi\big(X_1,\ldots,\widetilde{X}_l,\ldots,X_h,\ldots,X_n\big)\,\mathrm{d}v_l. \tag{18}$$

This equality will be labelled $(18)_h$.

(a) To start the inductive process, we must show that

$$\int_X \varphi \,\mathrm{d}v = \frac{d_1}{a'_1} \int_{\widetilde{X}_1 \leqslant 1} \varphi\big(\widetilde{X}_1, X_2,\ldots,X_n\big)\,\mathrm{d}v_1.$$

Since we work in parametrization $\psi_1 : \mathbb{C}_*^m \to V$, which covers a'_1-times V, if

$$\Omega = \big\{\big(\widetilde{X}_1,\ldots,X_n\big); \widetilde{X}_1 \leqslant 1\big\}$$

and

$$\Omega' = \big\{\big(\widetilde{X}_1,\ldots,X_n\big); x_1 \geqslant 1 \text{ and } x_2,\ldots,x_{b_1} \leqslant x_1\big\},$$

we have

$$a_1' \int\limits_X \varphi \, \mathrm{d}v = \int\limits_{\mathbb{C}_*^m} \varphi\big(\widetilde{X}_1, \ldots, X_n\big) \, \mathrm{d}v_1 = \int\limits_{\Omega} \varphi\big(\widetilde{X}_1, \ldots, X_n\big) \, \mathrm{d}v_1 + (d_1 - 1) \int\limits_{\Omega'} \varphi\big(\widetilde{X}_1, \ldots, X_n\big) \, \mathrm{d}v_1$$

(using $\sigma_{1,j}$-invariance of φ, for $2 \leqslant j \leqslant d_1 - 1$). Next, we take into account that $\sigma_{0,1}$ is an involutive isometry of (X, g) which keeps invariant V. In parametrization ψ_1, it exchanges the subsets Ω and Ω', since according to 2.2,

$$\sigma_{0,1}\big(\widetilde{Z}_1, Z_2, \ldots, Z_n\big) = \left(1, \frac{1}{z_1}, \frac{z_2}{z_1}, \ldots, \frac{z_{b_1}}{z_1}; \frac{Z_2}{z_1^{a_1/a_2}}; \ldots; \frac{Z_n}{z_1^{a_1/a_n}}\right).$$

Hence, $\int_{\Omega'} \varphi_1 \, \mathrm{d}v_1 = \int_{\Omega} \varphi_1 \, \mathrm{d}v_1$ and $(18)_1$ follows.

(b) Suppose now that $(18)_{h-1}$ is true for some $h = \{2, \ldots, n\}$. If $1 \leqslant l \leqslant h-1$, $\delta_h = b_{h-1} + 1$ and

$$A_{l,h} = \Big\{\big(Z_1, \ldots, \widetilde{Z}_l, \ldots, Z_n\big);\, X_1, \ldots, \widetilde{X}_l, \ldots, X_{h-1} \leqslant 1, x_{\delta_h} \geqslant 1 \text{ and } \max_{j \in I_h} x_j = x_{\delta_h}\Big\},$$

by virtue of the $\sigma_{\delta_h, j}$-invariance of φ for $j \in I_h$, we get

$$\int\limits_{X_1, \ldots, \widetilde{X}_l, \ldots, X_{h-1} \leqslant 1} \varphi_l \, \mathrm{d}v_l = \int\limits_{X_1, \ldots, \widetilde{X}_l, \ldots, X_h \leqslant 1} \varphi_l \, \mathrm{d}v_l + d_h \int\limits_{A_{l,h}} \varphi_l \, \mathrm{d}v_l.$$

Consequently, thanks to $(18)_{h-1}$,

$$\int\limits_X \varphi \, \mathrm{d}v = \sum_{l=1}^{h-1} \frac{d_l}{a_l'} \int\limits_{X_1, \ldots, \widetilde{X}_l, \ldots, X_h \leqslant 1} \varphi_l \, \mathrm{d}v_l + \sum_{l=1}^{h-1} \frac{d_l d_h}{a_l'} \int\limits_{A_{l,h}} \varphi_l \, \mathrm{d}v_l$$

and, to obtain $(18)_h$, we have to prove that

$$\sum_{l=1}^{h-1} \frac{d_l}{a_l'} \int\limits_{A_{l,h}} \varphi_l \, \mathrm{d}v_l = \frac{1}{a_h'} \int\limits_{X_1, \ldots, X_{h-1}, \widetilde{X}_h \leqslant 1} \varphi_h \, \mathrm{d}v_h. \tag{19}$$

Notice that $\frac{1}{a_l'} \int_{A_{l,h}} \varphi_l \, \mathrm{d}v_l$ is the integral of φ, written in terms of parametrization ψ_l, over the subset

$$\widetilde{A}_{l,h} = \big\{\big([Z_1], \ldots, [\widetilde{Z}_l], \ldots, [Z_n], \big[Z_1^{a_1}, \ldots, \widetilde{Z}_l^{a_l}, \ldots, Z_n^{a_n}\big]\big) \in X;\, \big(Z_1, \ldots, \widetilde{Z}_l, \ldots, Z_n\big) \in A_{l,h}\big\},$$

which is also described as

$$\begin{aligned}
\widetilde{A}_{l,h} \ &= \ \Bigg\{\Bigg(\left[\frac{Z_1}{z_{\delta_h}^{a_h/a_1}}\right], \ldots, \left[\frac{\widetilde{Z}_l}{z_{\delta_h}^{a_h/a_l}}\right], \ldots, \left[\frac{Z_h}{z_{\delta_h}}\right], \ldots, \left[\frac{Z_n}{z_{\delta_h}^{a_h/a_n}}\right], \big[Z_1^{a_1}, \ldots, \widetilde{Z}_l^{a_l}, \ldots, Z_n^{a_n}\big]\Bigg) \in X; \\
&\qquad \big(Z_1, \ldots, \widetilde{Z}_l, \ldots, Z_n\big) \in A_{l,h}\Bigg\} \\
&= \ \Big\{\big([Z_1'], \ldots, [\widetilde{Z}_h'], \ldots, [Z_n'], \big[{Z'}_1^{a_1}, \ldots, {\widetilde{Z}'}{}_h^{a_h}, \ldots, {Z'}_n^{a_n}\big]\big) \in X;\ X_1', \ldots, X_{h-1}', \widetilde{X}_h' \leqslant 1, \max_{0 \leqslant j \leqslant \delta_h - 1} x_j = x_{\delta_l}\Big\}.
\end{aligned}$$

Thus, writing $\int_{\widetilde{A}_{l,h}} \varphi \, \mathrm{d}v$ in terms of parametrization ψ_h yields

$$\begin{aligned}
\frac{d_l}{a_l'} \int\limits_{A_{l,h}} \varphi_l \, \mathrm{d}v_l \ &= \ \frac{d_l}{a_h'} \int\limits_{X_1, \ldots, \widetilde{X}_h \leqslant 1, \max(x_0, \ldots, x_{\delta_h - 1}) = x_{\delta_l}} \varphi\big(Z_1, \ldots, \widetilde{Z}_h, \ldots, Z_n\big) \, \mathrm{d}v_h \\
&= \ \frac{1}{a_h'} \int\limits_{X_1, \ldots, \widetilde{X}_h \leqslant 1, \max(x_0, \ldots, x_{\delta_h - 1}) = x_k \text{ for some } k \in I_l} \varphi_h \, \mathrm{d}v_h,
\end{aligned}$$

where the last equality is obtained thanks to the $\sigma_{\delta_l, j}$-invariance of φ for $j \in I_l$.

Finally, by summation over $l = 1, \ldots, h-1$, we get the requested equality (19).

3.2. End of the proof of Theorem 2

The proof of Theorem 2 uses an invariant introduced by Tian [26] and defined as follows:

$$\alpha_G(X)=\sup\left\{\alpha>0;\ \exists C \text{ such that } \forall\varphi\in A_G,\ \int_X \exp(-\alpha\varphi)\,\mathrm{d}v\leqslant C\exp\left(\frac{-\alpha}{V}\int_X\varphi\,\mathrm{d}v\right)\right\}.$$

Using a strategy initiated by Aubin in the fundamental work [2], and an inequality involving the integral of the exponential of plurisubharmonic functions due to type Bombieri [14], Skoda [25] and Hörmander [18], one shows (see Aubin [5], Tian [26]) that a lower bound of this invariant gives a C^0 estimate of the solutions of the family of Monge–Ampère equations

$$\log M(\varphi)=-t\varphi+f,\quad t>0,$$

where f is the geometric datum given by $\mathrm{Ricci}(\omega)-\omega=\frac{\mathrm{i}}{2\pi}\partial\bar\partial f$,

$$M(\varphi)=\det\bigl(\delta_\lambda^\mu+\nabla_\lambda^\mu\varphi\bigr)_{\lambda,\mu\in\{1,\dots,m\}},\quad\text{and}\quad \omega=\frac{\mathrm{i}}{2\pi}g_{\lambda\bar\mu}\,\mathrm{d}z^\lambda\wedge\mathrm{d}z^{\bar\mu}\in C_1(X).$$

In fact if $\alpha_G>tm/(m+1)$, one has the required C^0 estimate for the previous equation. By higher order a priori estimates obtained by Aubin [3], the C^0 estimate yields a solution of the Monge–Ampère equation. If we reach $t=1$, the manifold admits a Kähler–Einstein metric given by

$$g'_{\lambda\bar\mu}=g_{\lambda\bar\mu}+\partial_{\lambda\bar\mu}\varphi.$$

And now, let us give the proof of Theorem 2. Let $0<\alpha<1$. Given any G-invariant, g-admissible function $\varphi\in C^\infty(X)$ such that $\int_X\varphi\,\mathrm{d}v=0$, we shall bound from above, independently of φ, the integral $\int_X\exp(-\alpha\varphi)\,\mathrm{d}v$. Consequently, Tian's invariant $\alpha_G(X)$ is $\geqslant 1$, which proves the existence of an Einstein–Kähler metric on X.

We work in parametrization ψ_1 of V. Thanks to Lemma 2, we have to bound from above

$$\int_D \mathrm{e}^{-\alpha\varphi}\,\mathrm{d}v,$$

where D is identified to $\{(1,z_1,\dots,z_m);\ 0<x_1,\dots,x_m\leqslant 1\}$. First, according to Proposition 1, the volume element of the metric $\psi_1^*(g)$ is such that

$$\mathrm{d}v\leqslant C\prod_{h=2}^{n-1}\left(\sum_{j\in I_h}x_j\right)^{a-d}\mathrm{d}x_1\cdots\mathrm{d}x_m\quad\text{on } D.$$

Next, since $K=\log(T^n\prod_{h=1}^n t_h^{d-a})$ is the potential of g in the parametrization ψ_1 of V, we have

$$\mathrm{e}^{\alpha K}\leqslant C\prod_{h=2}^n t_h^{\alpha(d-a)}\quad\text{on } D.$$

Then, according to Lemma 1, $-(K+\varphi)(1,\dots,1)\leqslant\text{Const.}$ and, thanks to Proposition 4, we get

$$\begin{aligned}\int_D\mathrm{e}^{-\alpha\varphi}\,\mathrm{d}v&=\int_D\mathrm{e}^{-\alpha(K+\varphi)+\alpha K}\,\mathrm{d}v\leqslant\mathrm{e}^{-\alpha(K+\varphi)(1,\dots,1)}\int_D\frac{\mathrm{e}^{\alpha K}}{\prod_{j=1}^n x_j^\alpha}\,\mathrm{d}v\\&\leqslant C\int_D\left(\prod_{h=2}^n t_h^{\alpha(d-a)+(a-d)}\right)(x_1\cdots x_m)^{-\alpha}\,\mathrm{d}x_1\cdots\mathrm{d}x_m.\end{aligned}$$

Hence, since $\alpha<1$,

$$\begin{aligned}\int_D\mathrm{e}^{-\alpha\varphi}\,\mathrm{d}v&\leqslant C\int_{0<x_d,\dots,x_m\leqslant 1}\left(\prod_{h=2}^n t_h^{(\alpha-1)(d-a)}\right)(x_d\cdots x_m)^{-\alpha}\,\mathrm{d}x_d\cdots\mathrm{d}x_m\\&=C\left\{\int_{0<y_1,\dots,y_d\leqslant 1}\frac{\mathrm{d}y_1\cdots\mathrm{d}y_d}{(y_1+\cdots+y_d)^{(1-\alpha)(d-a)}(y_1\cdots y_d)^\alpha}\right\}^{n-1}=C\left\{\int_0^1\frac{r^{d-1}}{r^{(1-\alpha)(d-a)}r^{\alpha d}}\,\mathrm{d}r\right\}^{n-1}\end{aligned}$$

$$= C\left(\int_0^1 \frac{\mathrm{d}r}{r^{1+(\alpha-1)a}}\right)^{n-1} = \text{Const.},$$

because $(\alpha-1)a < 0$. Thus, the requested bound is obtained.

Appendix A

Proof of Proposition 1. (1) First, we explicit the terms of the matrix M. We have

$$\partial_{\overline{\mu}} \log T = \frac{\alpha_\mu z_\mu^{\alpha_\mu} z_{\overline{\mu}}^{\alpha_\mu - 1}}{T},$$

and, if we denote the Kronecker symbols by $\delta_{\lambda\mu}$,

$$\partial_{\lambda\overline{\mu}} \log T = \frac{\alpha_\lambda^2 x_\lambda^{\alpha_\lambda-1}}{T}\delta_{\lambda\mu} - \frac{\alpha_\lambda x_\lambda^{\alpha_\lambda-1} z_{\bar{\lambda}} \alpha_\mu x_\mu^{\alpha_\mu-1} z_\mu}{T^2};$$

on the other hand, for $\lambda, \mu \in J_h$,

$$\partial_{\lambda\overline{\mu}} \log t_h = \frac{\delta_{\lambda\mu}}{t_h} - \frac{\bar{z}_\lambda z_\mu}{t_h^2}.$$

Thus, we can write $M = D + P + Q$, with $D = \operatorname{diag}(\delta_\lambda)_{1\leqslant\lambda\leqslant m}$ a diagonal matrix, $P = (p_{\lambda\mu})_{1\leqslant\lambda,\mu\leqslant m}$ a direct sum of rank one matrices, and $Q = (q_{\lambda\mu})_{1\leqslant\lambda,\mu\leqslant m}$ a matrix of rank one, given by

$$D = \bigoplus_{h=1}^{n} \operatorname{diag}\left(\frac{d_h - a_h}{t_h} + \frac{n a_h^2 x_\lambda^{a_h-1}}{T}\right)_{\lambda\in J_h}, \qquad P = \bigoplus_{h=1}^{n} -\frac{d_h - a_h}{t_h^2}(\bar{z}_\lambda z_\mu)_{\lambda,\mu\in J_h} \quad \text{and} \quad Q = -\frac{n}{T^2}\big(\overline{V} v_\mu\big)_{1\leqslant\mu\leqslant m},$$

where $v_\mu = \alpha_\mu x_\mu^{\alpha_\mu-1} z_\mu$ and ${}^t V = (v_1, \dots, v_m)$. If D_μ, P_μ, Q_μ denote the columns of D, P, Q of index μ, since any Q_μ is colinear to V, we write

$$\det M = \varDelta_1 + \varDelta_2 \tag{A.1}$$

with

$$\varDelta_1 = \det(D_1 + P_1, \dots, D_m + P_m)$$

and

$$\varDelta_2 = \sum_{\mu=1}^{m} \det(D_1 + P_1, \dots, D_{\mu-1} + P_{\mu-1}, Q_\mu, D_{\mu+1} + P_{\mu+1}, \dots, D_m + P_m).$$

Taking into account the decomposition by blocks of $D + P$, and setting $B_h = (\operatorname{diag}\delta_\lambda)_{\lambda\in J_h} + (p_{\lambda\mu})_{\lambda,\mu\in J_h}$, we have

$$\varDelta_1 = \prod_{h=1}^{n} \det B_h. \tag{A.2}$$

Now, if a matrix S of order d is sum of $\operatorname{diag}(r_1, \dots, r_d)$ and of the rank one matrix $(b_1 W, \dots, b_d W)$, where ${}^t W = (w_1, \dots, w_d)$, then

$$\det S = r_1 \cdots r_d + \sum_{l=1}^{d} r_1 \cdots r_{l-1} b_l w_l r_{l+1} \cdots r_d.$$

Hence, since the rank of $(p_{\lambda\mu})_{\lambda,\mu\in J_h}$ is equal to one,

$$\det B_h = \prod_{\lambda\in J_h} \delta_\lambda + \sum_{\lambda\in J_h}\left(\prod_{\nu\in J_h, \nu\neq\lambda} \delta_\nu\right) p_{\lambda\lambda} = \left(\prod_{\lambda\in J_h} \delta_\lambda\right)\left(1 + \sum_{\lambda\in J_h} \frac{p_{\lambda\lambda}}{\delta_\lambda}\right)$$

and, using the explicit values of δ_λ and $p_{\lambda\lambda}$,

$$\det B_h = \frac{1}{(t_h T)^{d_h^*}} \prod_{\lambda\in J_h} \left[(d_h - a_h)T + na_h^2 x_\lambda^{a_h-1} t_h\right]\left[1 - \frac{(d_h-a_h)T}{t_h} \sum_{\lambda\in J_h} \frac{x_\lambda}{(d_h-a_h)T + na_h^2 x_\lambda^{a_h-1} t_h}\right], \tag{A.3}$$

with $d_1^* = d_1 - 1$ and $d_h^* = d_h$ if $h \geqslant 2$.

Let us now study δ_2 which we expand as the sum of m terms. Collecting the terms which belong to the same subset J_h, we write

$$\Delta_2 = \sum_1^n \det B_1 \cdots \det B_{h-1}\left(\sum_{\mu\in J_h} \det \Gamma_\mu\right) \det B_{h+1} \cdots \det B_n. \tag{A.4}$$

For $\mu \in J_h$, Γ_μ is the matrix of order d_h^* whose column $(\Gamma_\mu)_\mu$ is the projection of Q_μ on $\mathbb{C}^{d_h^*}$ in the decomposition $\mathbb{C}^m = \bigoplus_{h=1}^n \mathbb{C}^{d_h^*}$, and whose column $(\Gamma_\mu)_\nu$, for $\nu \in J_h$, $\nu \neq \mu$, is the projection $\widetilde{D}_\nu + \widetilde{P}_\nu$ of $D_\nu + P_\nu$ on $\mathbb{C}^{d_h^*}$ (which could be identified with $D_\nu + P_\nu$, since the components of $D_\nu + P_\nu$ of indices belonging to $\bigcup_{i\neq h} J_i$ are equal to zero).

We want to compute $\Gamma_{(h)} = \sum_{\mu\in J_h} \det \Gamma_\mu$. To simplify the notations, we suppose that $h = 1$; in the summations which occur, all the indices μ, ν, ρ belong to $J_1 = \{1, \ldots, d_1^*\}$. We have

$$\begin{aligned}\Gamma_{(1)} &= \sum_{\mu\in J_1} \det \Gamma_\mu = \sum_\mu \det\bigl(\widetilde{D}_1 + \widetilde{P}_1, \ldots, \widetilde{D}_{\mu-1} + \widetilde{P}_{\mu-1}, \widetilde{Q}_\mu, \widetilde{D}_{\mu+1} + \widetilde{P}_{\mu+1}, \ldots, \widetilde{D}_{d_1^*} + \widetilde{P}_{d_1^*}\bigr)\\ &= \mathrm{I} + \mathrm{II} + \mathrm{III},\end{aligned} \tag{A.5}$$

where, since all the vectors $\widetilde{P}_\nu$ are parallel,

$$\begin{aligned}\mathrm{I} &= \sum_\mu \det\bigl(\widetilde{D}_1, \ldots, \widetilde{D}_{\mu-1}, \widetilde{Q}_\mu, \widetilde{D}_{\mu+1}, \ldots, \widetilde{D}_{d_1^*}\bigr),\\ \mathrm{II} &= \sum_\mu \sum_{\nu<\mu} \det\bigl(\widetilde{D}_1, \ldots, \widetilde{D}_{\nu-1}, \widetilde{P}_\nu, \ldots, \widetilde{D}_\rho, \ldots, \widetilde{Q}_\mu, \widetilde{D}_{\mu+1}, \ldots, \widetilde{D}_{d_1^*}\bigr),\\ \mathrm{III} &= \sum_\mu \sum_{\mu<\nu} \det\bigl(\widetilde{D}_1, \ldots, \widetilde{D}_{\mu-1}, \widetilde{Q}_\mu, \ldots, \widetilde{D}_\rho, \ldots, \widetilde{P}_\nu, \widetilde{D}_{\nu+1}, \ldots, \widetilde{D}_{d_1^*}\bigr).\end{aligned}$$

Recall that the element δ_ν of the diagonal matrix D is given by

$$\delta_\nu = \frac{d_1 - a_1}{t_1} + \frac{na_1^2 x_\nu^{a_1-1}}{T}.$$

First,

$$\mathrm{I} = -\sum_\mu \left[\frac{na_1^2 x_\mu^{2a_1-1}}{T^2} \prod_{\nu\neq\mu} \delta_\nu\right]. \tag{A.6}$$

On the other hand,

$$\mathrm{II} + \mathrm{III} = \sum_\mu \sum_{\nu<\mu} \left(\prod_{\rho\neq\mu,\nu} \delta_\rho\right) \gamma_{\nu\mu}^{(\mu)} + \sum_\mu \sum_{\mu<\nu} \left(\prod_{\rho\neq\mu,\nu} \delta_\rho\right) \gamma_{\mu\nu}^{(\mu)}.$$

For $\nu < \mu$, the quantity $\gamma_{\nu\mu}^{(\mu)}$ is defined by

$$\gamma_{\nu\mu}^{(\mu)} = \det\begin{pmatrix} -\frac{d_1-a_1}{t_1^2} \bar{z}_\nu z_\nu & -\frac{n}{T^2} a_1 x_\nu^{a_1-1} \bar{z}_\nu a_1 x_\mu^{a_1-1} z_\mu \\ -\frac{d_1-a_1}{t_1^2} \bar{z}_\mu z_\nu & -\frac{n}{T^2} a_1 x_\mu^{a_1-1} \bar{z}_\mu a_1 x_\mu^{a_1-1} z_\mu \end{pmatrix} = n(d_1 - a_1) a_1^2 \frac{x_\nu x_\mu^{a_1}\bigl(x_\mu^{a_1-1} - x_\nu^{a_1-1}\bigr)}{t_1^2 T^2},$$

and, for $\mu < \nu$,

$$\gamma_{\mu\nu}^{(\mu)} = \det\begin{pmatrix} -\frac{n}{T^2} a_1 x_\mu^{a_1-1} \bar{z}_\mu a_1 x_\mu^{a_1-1} z_\mu & -\frac{d_1-a_1}{t_1^2} \bar{z}_\mu z_\nu \\ -\frac{n}{T^2} a_1 x_\nu^{a_1-1} \bar{z}_\nu a_1 x_\mu^{a_1-1} z_\mu & -\frac{d_1-a_1}{t_1^2} \bar{z}_\nu z_\nu \end{pmatrix} = n(d_1 - a_1) a_1^2 \frac{x_\mu^{a_1} x_\nu\bigl(x_\mu^{a_1-1} - x_\nu^{a_1-1}\bigr)}{t_1^2 T^2}.$$

Thus,

$$\mathrm{II} + \mathrm{III} = \sum_{\mu<\nu} \left(\prod_{\rho\neq\mu,\nu} \delta_\rho\right)\bigl(\gamma_{\mu\nu}^{(\mu)} + \gamma_{\mu\nu}^{(\nu)}\bigr) = \frac{n(d_1-a_1)a_1^2}{t_1^2 T^2} \sum_{\mu<\nu} \left(\prod_{\rho\neq\mu,\nu} \delta_\rho\right) x_\mu x_\nu \bigl(x_\mu^{a_1-1} - x_\nu^{a_1-1}\bigr)^2,$$

since, for $\mu < \nu$,

$$\begin{aligned}\gamma_{\mu\nu}^{(\mu)} + \gamma_{\mu\nu}^{(\nu)} &= \frac{n(d_1 - a_1)a_1^2}{t_1^2 T^2}\Big[x_\mu^{a_1} x_\nu\big(x_\mu^{a_1-1} - x_\nu^{a_1-1}\big) + x_\mu x_\nu^{a_1}\big(x_\nu^{a_1-1} - x_\mu^{a_1-1}\big)\Big] \\ &= \frac{n(d_1 - a_1)a_1^2}{t_1^2 T^2} x_\mu x_\nu \big(x_\mu^{a_1-1} - x_\nu^{a_1-1}\big)^2.\end{aligned}$$

Taking into account (A.6), (A.7) and (A.5), we obtain, for any $h = 1, \ldots, n$, the value of $\Gamma_{(h)}$:

$$\begin{aligned}\Gamma_{(h)} = \sum_{\mu\in J_h} \det \Gamma_\mu &= -\sum_{\mu\in J_h} \frac{na_h^2 x_\mu^{2a_h-1}}{T^2}\Bigg(\prod_{\nu\in J_h,\nu\neq\mu} \delta_\nu\Bigg) \\ &\quad + \frac{n(d_h - a_h)a_h^2}{t_h^2 T^2} \sum_{\mu,\nu\in J_h,\mu<\nu} \Bigg(\prod_{\rho\in J_h,\rho\neq\mu,\nu} \delta_\rho\Bigg) x_\mu x_\nu \big(x_\mu^{a_h-1} - x_\nu^{a_h-1}\big)^2,\end{aligned} \tag{A.7}$$

recalling that $\delta_\lambda = (d_h - a_h)/t_h + na_h^2 x_\lambda^{a_h-1}/T$ if $\lambda \in J_h$.

(2) We now look for any upper bound of $\det M$ on the cube

$$D = \big\{Z = (z_k)_{1\leqslant k\leqslant m} \in \mathbb{C}^m;\ 0 < x_k \leqslant 1 \text{ for all } k\big\}.$$

Let us study $\det B_h$ defined in (A.3). We write $\det B_h = B_h' B_h''$, with

$$B_h' = \prod_{\lambda\in J_h}\left(\frac{d_h - a_h}{t_h} + \frac{na_h^2 x_\lambda^{a_h-1}}{T}\right)$$

and, if $b_h = na_h^2/(d_h - a_h)$,

$$B_h'' = 1 - \frac{1}{t_h}\sum_{\lambda\in J_h} x_\lambda \left(1 + b_h \frac{x_\lambda^{a_h-1} t_h}{T}\right)^{-1}.$$

Since $0 < x_\lambda \leqslant 1$ on D, and according to the definitions of t_h and T (in particular, T and t_1 are > 1 on D), we have

$$\left[\frac{d_h - a_h}{t_h}\right]^{d_h^*} < B_h' < \left[\frac{(m+1)(d_h - a_h) + na_h^2 d_h}{t_h}\right]^{d_h^*}$$

and

$$1 - \frac{1}{t_h}\sum_{\lambda\in J_h} x_\lambda = 0 < B_h'' < 1 - \left(1 + \frac{b_h t_h}{T}\right)^{-1} < b_h t_h.$$

To get a more precise upper bound of B_h'' when $h \geqslant 2$, since $x_\lambda \leqslant t_h = \sum_{\mu\in J_h} x_\mu$ if $\lambda \in J_h$, we write:

$$B_h'' = 1 - \frac{1}{t_h}\sum_{\lambda\in J_h}\left(x_\lambda - b_h\frac{x_\lambda^{a_h} t_h}{T} + \mathrm{o}\big(x_\lambda^{a_h} t_h\big)\right) = \sum_{\lambda\in J_h} \frac{b_h x_\lambda^{a_h}}{T}\big(1 + \mathrm{o}\big(x_\lambda^{a_h}\big)\big) \leqslant \text{Const.} \times t_h^{a_h}.$$

Hence, there exist positive constants C' and C'' such that, on D,

$$C' \leqslant \det B_1 \leqslant C'', \qquad \frac{C'}{t_h^{d_h-a_h}} \leqslant \det B_h \leqslant \frac{C''}{t_h^{d_h-a_h}} \quad \text{if } h \geqslant 2$$

and, consequently,

$$0 < \prod_{h=1}^n \det B_h \leqslant \frac{\text{Const.}}{\prod_{h=2}^n t_h^{d_h-a_h}}. \tag{A.8}$$

Next, we examine, on D, $\Gamma_{(h)}$ as defined in (A.7). It is clear that $|\Gamma_{(1)}| \leqslant$ Const. Suppose $h \geqslant 2$. Since for $\lambda, \mu, \nu \in I_h$,

$$\frac{\text{Const.}}{t_h} \leqslant \delta_\lambda = \frac{(d_h - a_h)T + na_h^2 x_\lambda^{a_h-1} t_h}{t_h T} \leqslant \frac{\text{Const.}}{t_h}$$

and, if $x_\mu \geqslant x_\nu$,

$$\frac{x_\mu x_\nu}{t_h^2}\left(x_\mu^{a_h-1} - x_\nu^{a_h-1}\right)^2 \leqslant \frac{x_\mu^2 x_\mu^{2a_h-2}}{t_h^2} \leqslant t_h^{2a_h-2},$$

we see that

$$\sum_{\mu\in J_h} \frac{n a_h^2 x_\mu^{2a_h-1}}{T^2}\left(\prod_{\nu\in J_h,\nu\neq\mu}\delta_\nu\right) \leqslant \text{Const.}\frac{t_h^{2a_h-1}}{t_h^{d_h-1}} = \frac{\text{Const.}}{t_h^{d_h-2a_h}}$$

and

$$\frac{n(d_h-a_h)a_h^2}{t_h^2 T^2}\sum_{\mu,\nu\in J_h,\mu<\nu}\left(\prod_{\rho\in J_h,\rho\neq\mu,\nu}\delta_\rho\right)x_\mu x_\nu\left(x_\mu^{a_h-1} - x_\nu^{a_h-1}\right)^2 \leqslant \text{Const.}\frac{t_h^{2a_h-2}}{t_h^{d_h-2}} = \frac{\text{Const.}}{t_h^{d_h-2a_h}}.$$

Thus, according to (A.7),

$$|\Gamma_{(h)}| \leqslant \frac{\text{Const.}}{t_h^{d_h-2a_h}} \leqslant \frac{\text{Const.}}{t_h^{d_h-a_h}}$$

and so

$$\left|\sum_{h=1}^{n} \det B_1\cdots\det B_{h-1}\Gamma_{(h)}\det B_{h+1}\cdots\det B_n\right| \leqslant \frac{\text{Const.}}{\prod_{h=2}^{n} t_h^{d_h-a_h}}. \tag{A.9}$$

If we take into account the value of $\det M$ given in the statement of the proposition, (A.8) and (A.9) yield to the upper bound of $\det M$ we were seeking. □

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